Simultaneous local exact controllability of bilinear Schrödinger equations

Morgan Morancey *†

November 7, 2012

Abstract

We consider N independent quantum particles, in an infinite square potential well and an external electric field. These particles are modelled by a system of linear Schrödinger equations on a bounded interval. This is a bilinear control system in which the state is the N-tuple of wave functions. The control is the real amplitude of the electric field. For $N = 1$, Beauchard and Laurent proved local exact controllability around the ground state in arbitrary time. We prove, under an extra generic assumption, that their result does not hold in small time if $N \geq 2$. Still, for $N = 2$, we prove that local controllability holds either in arbitrary time up to a global phase or exactly up to a global delay. We also prove that for $N \geq 3$, local controllability does not hold in small time even up to a global phase. Finally, for $N = 3$, we prove that local controllability holds up to a global phase and a global delay.

Keywords : bilinear control, Schrödinger equation, simultaneous control, return method, non controllability.

1 Introduction

1.1 Main results

As proposed in [44], we consider a quantum particle in a one dimensional infinite square potential well submitted to an external electric field. The evolution of the wave function ψ is given by the following Schrödinger equation

$$\begin{cases} i\partial_t \psi = -\partial^2_{xx}\psi - u(t)\mu(x)\psi, & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \end{cases} \tag{1.1}$$

where $\mu \in H^3((0,1),\mathbb{R})$ is the dipolar moment and $u : t \in (0,T) \mapsto \mathbb{R}$ is the amplitude of the external electric field. This is a bilinear control system in which the state ψ lives on a sphere of $L^2((0,1),\mathbb{C})$.

*CMLS UMR 7640, Ecole Polytechnique, 91128 Palaiseau, FRANCE. email: Morgan.Morancey@cmla.ens-cachan.fr

†The author was partially supported by the "Agence Nationale de la Recherche" (ANR), Projet Blanc EMAQS number ANR-2011-BS01-017-01 and by CMLA UMR 8536, ENS Cachan, 61 avenue du Président Wilson, 94235 Cachan, FRANCE.

Let us consider a set of N identical and independent particles submitted to a single external electric field. Neglecting entanglement they are represented by the system

$$\begin{cases} i\partial_t \psi^j = -\partial^2_{xx}\psi^j - u(t)\mu(x)\psi^j, & (t,x) \in (0,T)\times(0,1),\ j\in\{1,\dots,N\}, \\ \psi^j(t,0) = \psi^j(t,1) = 0, & t\in(0,T),\ j\in\{1,\dots,N\}. \end{cases} \tag{1.2}$$

Before going into details, let us set some notations. In this paper, $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $L^2((0,1),\mathbb{C})$ i.e.

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x$$

and $\mathcal{S}$ denotes the unit sphere of $L^2((0,1),\mathbb{C})$. We consider the operator A defined by

$$\mathcal{D}(A) := H^2 \cap H^1_0((0,1),\mathbb{C}), \quad A\varphi := -\partial^2_{xx}\varphi.$$

Its eigenvalues and eigenvectors are

$$\lambda_k := (k\pi)^2, \quad \varphi_k(x) := \sqrt{2}\sin(k\pi x), \quad \forall k\in\mathbb{N}^*.$$

The family $(\varphi_k)_{k\in\mathbb{N}^*}$ is an Hilbert basis of $L^2((0,1),\mathbb{C})$. The eigenstates are defined by

$$\Phi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \quad (t,x)\in\mathbb{R}^+\times(0,1),\ k\in\mathbb{N}^*.$$

Any N-tuple of eigenstates is solution of system (1.2) with control $u\equiv 0$. Finally, we define the spaces

$$H^s_{(0)}((0,1),\mathbb{C}) := \mathcal{D}(A^{s/2}),\ \forall s>0,$$

endowed with the norm

$$||\cdot||_{H^s_{(0)}} := \left(\sum_{k=1}^{+\infty} |k^s\langle\,\cdot\,,\varphi_k\rangle|\right)^{1/2}$$

and

$$h^s(\mathbb{N}^*,\mathbb{C}) := \left\{ a = (a_k)_{k\in\mathbb{N}^*}\in\mathbb{C}^{\mathbb{N}^*}\ ;\ \sum_{k=1}^{+\infty}|k^s a_k|^2 < +\infty \right\}$$

endowed with the norm

$$||a||_{h^s} := \left(\sum_{k=1}^{+\infty}|k^s a_k|^2\right)^{1/2}.$$

Our goal is to control simultaneously the particles modelled by (1.2) with initial conditions

$$\psi^j(0,x) = \varphi_j(x), \quad x\in(0,1),\ j\in\{1,\dots,N\}, \tag{1.3}$$

locally around $\big(\Phi_1,\dots,\Phi_N\big)$ using a single control.

The case $N=1$ of a single equation was studied, in this setting, in [6, Theorem 1] by Beauchard and Laurent. They proved exact controllability, in $H^3_{(0)}$, in arbitrary time, locally around Φ_1. Their proof relies on the linear test, the inverse mapping theorem and a regularizing effect. We prove that this result cannot be extended to the case $N=2$.

In the spirit of [6], we assume the following hypothesis.

Hypothesis 1.1. The dipolar moment $\mu \in H^3((0,1),\mathbb{R})$ is such that there exists $c>0$ satisfying

$$|\langle \mu\varphi_j, \varphi_k\rangle| \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*, \forall j \in \{1,\dots,N\}.$$

Remark 1.1. In the same way as in [6, Proposition 16], one may prove that Hypothesis 1.1 holds generically in $H^3((0,1),\mathbb{R})$.

Using [6, Theorem 1], Hypothesis 1.1 implies that the j^{th} equation of system (1.2) is locally controllable.

Hypothesis 1.2. The dipolar moment $\mu \in H^3((0,1),\mathbb{R})$ is such that

$$\mathcal{A} := \langle \mu\varphi_1, \varphi_1\rangle\langle (\mu')^2\varphi_2, \varphi_2\rangle - \langle \mu\varphi_2, \varphi_2\rangle\langle (\mu')^2\varphi_1, \varphi_1\rangle \neq 0.$$

Remark 1.2. For example, $\mu(x) := x^3$ satisfies both Hypothesis 1.1 and 1.2. As in [6, Proposition 16], one may prove that Hypothesis 1.1 and 1.2 hold simultaneously generically in $H^3((0,1),\mathbb{R})$.

Remark 1.3. Hypothesis 1.2 implies that there exists $j \in \{1,2\}$ such that $\langle \mu\varphi_j, \varphi_j\rangle \neq 0$. Without loss of generality, when Hypothesis 1.2 is assumed to hold, one should consider that $\langle \mu\varphi_1, \varphi_1\rangle \neq 0$.

Theorem 1.1. *Let $N=2$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 1.2 hold. Let $\alpha \in \{-1,1\}$ be defined by $\alpha := sign(\mathcal{A}\langle \mu\varphi_1, \varphi_1\rangle)$. There exists $T_*>0$ and $\varepsilon>0$ such that for any $T<T_*$, for every $u \in L^2((0,T),\mathbb{R})$ with*

$$\left(\int_0^T \left|\int_0^t u(\tau)\mathrm{d}\tau\right|^2 \mathrm{d}t\right)^{1/2} < \varepsilon,$$

the solution of system (1.2)-(1.3) satisfies

$$\big(\psi^1(T), \psi^2(T)\big) \neq \Big(\Phi_1(T), \Big(\sqrt{1-\delta^2} + i\alpha\delta\Big)\,\Phi_2(T)\Big), \; \forall \delta > 0.$$

Thus, under Hypothesis 1.2, simultaneous controllability does not hold for (ψ^1,ψ^2) around (Φ_1,Φ_2) in small time with small controls. The smallness assumption on the control is in H^{-1} norm. This prevents from extending [6, Theorem 1] to the case $N \geq 2$. However, when modelling a quantum particle, the global phase is physically meaningless. Thus for any $\theta \in \mathbb{R}$ and $\psi^1, \psi^2 \in L^2((0,1),\mathbb{C})$, the states $e^{i\theta}(\psi^1,\psi^2)$ and (ψ^1,ψ^2) are physically equivalent. Working up to a global phase, we prove the following theorem.

Theorem 1.2. *Let $N=2$. Let $T>0$. Let $\mu \in H^3((0,1),\mathbb{C})$ satisfy Hypothesis 1.1 and $\langle \mu\varphi_1, \varphi_1\rangle \neq \langle \mu\varphi_2, \varphi_2\rangle$. There exists $\theta \in \mathbb{R}$, $\varepsilon_0>0$ and a C^1 map*

$$\Gamma : \mathcal{O}_{\varepsilon_0} \to L^2((0,T),\mathbb{R})$$

where

$$\mathcal{O}_{\varepsilon_0} := \Big\{ \big(\psi_f^1, \psi_f^2\big) \in H^3_{(0)}((0,1),\mathbb{C})^2\,;\; \langle \psi_f^j, \psi_f^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^2 ||\psi_f^j - e^{i\theta}\Phi_j(T)||_{H^3_{(0)}} < \varepsilon_0 \Big\},$$

such that for any $\big(\psi_f^1, \psi_f^2\big) \in \mathcal{O}_{\varepsilon_0}$*, the solution of system (1.2) with initial condition (1.3) and control* $u = \Gamma\big(\psi_f^1, \psi_f^2\big)$ *satisfy*

$$(\psi^1(T), \psi^2(T)) = (\psi_f^1, \psi_f^2).$$

Remark 1.4. Notice that $\langle \psi^j(t), \psi^k(t)\rangle$ is an invariant quantity of system (1.2), for every $j, k \in \{1, \ldots, N\}$. Using the initial conditions (1.3), it comes that $\langle \psi^j(0), \psi^k(0)\rangle = \delta_{j=k}$. Thus, the condition $\langle \psi_f^j, \psi_f^k\rangle = \delta_{j=k}$ is not restrictive.

Remark 1.5. The same theorem holds with initial conditions close enough to (φ_1, φ_2) satisfying the constraints (see Remark 4.1 in Section 4.2).

Working in time large enough we can drop the global phase and prove the following theorem.

Theorem 1.3. *Let* $N = 2$*. Let* $\mu \in H^3((0,1), \mathbb{C})$ *satisfy Hypothesis 1.1 and* $4\langle \mu\varphi_1, \varphi_1\rangle - \langle \mu\varphi_2, \varphi_2\rangle \neq 0$*. There exists* $T^* > 0$ *such that, for any* $T \geq 0$*, there exists* $\varepsilon_0 > 0$ *a* C^1 *map*

$$\Gamma : \mathcal{O}_{\varepsilon_0, T} \to L^2((0, T^* + T), \mathbb{R})$$

where

$$\mathcal{O}_{\varepsilon_0,T} := \Big\{ \big(\psi_f^1, \psi_f^2\big) \in H^3_{(0)}((0,1), \mathbb{C})^2 \, ; \, \langle \psi_f^j, \psi_f^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{2} ||\psi_f^j - \phi_j(T)||_{H^3_{(0)}} < \varepsilon_0 \Big\},$$

such that for any $\big(\psi_f^1, \psi_f^2\big) \in \mathcal{O}_{\varepsilon_0,T}$*, the solution of system (1.2) with initial condition (1.3) and control* $u = \Gamma\big(\psi_f^1, \psi_f^2\big)$ *satisfy*

$$\big(\psi^1(T^* + T), \psi^2(T^* + T)\big) = \big(\psi_f^1, \psi_f^2\big).$$

Remark 1.6. Remark 1.5 is still valid in this case.

We now turn to the case $N = 3$. We prove that under an extra generic assumption, Theorem 1.2 cannot be extended to three particles. Assume the following hypothesis.

Hypothesis 1.3. The dipolar moment $\mu \in H^3((0,1), \mathbb{R})$ is such that

$$\begin{aligned}
\mathcal{B} := &\big(\langle \mu\varphi_3, \varphi_3\rangle - \langle \mu\varphi_2, \varphi_2\rangle\big)\langle (\mu')^2\varphi_1, \varphi_1\rangle \\
&+ \big(\langle \mu\varphi_1, \varphi_1\rangle - \langle \mu\varphi_3, \varphi_3\rangle\big)\langle (\mu')^2\varphi_2, \varphi_2\rangle \\
&+ \big(\langle \mu\varphi_2, \varphi_2\rangle - \langle \mu\varphi_1, \varphi_1\rangle\big)\langle (\mu')^2\varphi_3, \varphi_3\rangle \neq 0.
\end{aligned}$$

Remark 1.7. Hypothesis 1.3 implies that there exist $j, k \in \{1, 2, 3\}$ such that $\langle \mu\varphi_j, \varphi_j\rangle \neq \langle \mu\varphi_k, \varphi_k\rangle$. Without loss of generality, when Hypothesis 1.3 is assumed to hold, one should consider that $\langle \mu\varphi_1, \varphi_1\rangle \neq \langle \mu\varphi_2, \varphi_2\rangle$.

Remark 1.8. Again, Hypothesis 1.1 and 1.3 hold simultaneously generically in $H^3((0,1), \mathbb{R})$.

We prove the following theorem.

Theorem 1.4. *Let $N = 3$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 1.3 hold. Let $\beta \in \{-1,1\}$ be defined by $\beta = sign\big(\mathcal{B}(\langle\mu\varphi_2,\varphi_2\rangle - \langle\mu\varphi_1,\varphi_1\rangle)\big)$. There exists $T_* > 0$ and $\varepsilon > 0$ such that, for any $T < T_*$, for every $u \in L^2((0,T),\mathbb{R})$ with*

$$\left(\int_0^T \left|\int_0^t u(\tau)\mathrm{d}\tau\right|^2 \mathrm{d}t\right)^{1/2} < \varepsilon,$$

the solution of system (1.2)-(1.3) satisfies for every $\delta > 0$ and $\nu \in \mathbb{R}$,

$$\big(\psi^1(T),\psi^2(T),\psi^3(T)\big) \neq e^{i\nu}\left(\Phi_1(T),\Phi_2(T),\left(\sqrt{1-\delta^2}+i\beta\delta\right)\Phi_3(T)\right).$$

Thus, in small time, local exact controllability with small controls does not hold for $N \geq 3$, even up to a global phase. The next statement ensures that it holds up to a global phase and a global delay.

Theorem 1.5. *Let $N = 3$. Let $\mu \in H^3((0,1),\mathbb{C})$ satisfy Hypothesis 1.1 and $5\langle\mu\varphi_1,\varphi_1\rangle - 8\langle\mu\varphi_2,\varphi_2\rangle + 3\langle\mu\varphi_3,\varphi_3\rangle \neq 0$. There exists $\theta \in \mathbb{R}$, $T^* > 0$ such that, for any $T \geq 0$, there exists $\varepsilon_0 > 0$ and a C^1 map*

$$\Gamma : \mathcal{O}_{\varepsilon_0,T} \to L^2((0,T^*+T),\mathbb{R})$$

where

$$\mathcal{O}_{\varepsilon_0,T} := \Big\{\big(\psi_f^1,\psi_f^2,\psi_f^3\big) \in H^3_{(0)}((0,1),\mathbb{C})^3\,;\, \langle\psi_f^j,\psi_f^k\rangle = \delta_{j=k} \text{ and } \sum_{j=1}^3 ||\psi_f^j - e^{i\theta}\phi_j(T)||_{H^3_{(0)}} < \varepsilon_0\Big\},$$

such that for any $\big(\psi_f^1,\psi_f^2,\psi_f^3\big) \in \mathcal{O}_{\varepsilon_0,T}$, the solution of system (1.2) with initial condition (1.3) and control $u = \Gamma\big(\psi_f^1,\psi_f^2,\psi_f^3\big)$ satisfy $\big(\psi^1(T^*+T),\psi^2(T^*+T),\psi^3(T^*+T)\big) = \big(\psi_f^1,\psi_f^2,\psi_f^3\big)$.

Remark 1.9. Remark 1.5 is still valid in this case.

1.2 Heuristic

Contrarily to the case $N = 1$, the linearized system around a couple of eigenstate is not controllable when $N \geq 2$. Let us consider, for $N = 2$, the linearization of system (1.2) around (Φ_1,Φ_2)

$$\begin{cases} i\partial_t\Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (0,T)\times(0,1),\ j \in \{1,2\}, \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (0,T),\ j \in \{1,2\}, \\ \Psi^j(0,x) = 0, & x \in (0,1),\ j \in \{1,2\}. \end{cases}$$

For $j = 1,2$, straightforward computations lead to

$$\Psi^j(T) = i\sum_{k=1}^{+\infty}\langle\mu\varphi_j,\varphi_k\rangle\int_0^T v(t)e^{i(\lambda_k-\lambda_j)t}\mathrm{d}t\Phi_k(T).$$

Thus, thanks to Hypothesis 1.1, we could, by solving a suitable moment problem, control any directions $\langle\Psi^j(T),\Phi_k(T)\rangle$, for $k \geq 3$. The fact that

$$\langle\Psi^1(T),\Phi_2(T)\rangle + \overline{\langle\Psi^2(T),\Phi_1(T)\rangle} = 0,$$

come from the linearization of the invariant

$$\langle \psi^1(t), \psi^2(t)\rangle = \langle \psi^1_0, \psi^2_0\rangle, \quad \forall t \in (0,T),$$

and can be overcome (see Subsection 4.2). However, the fact that

$$\langle \mu\varphi_2, \varphi_2\rangle \langle \Psi^1(T), \Phi_1(T)\rangle = \langle \mu\varphi_1, \varphi_1\rangle \langle \Psi^2(T), \Phi_2(T)\rangle$$

is a strong obstacle to controllability (see Section 6).
In this situation, where a direction is lost at the first order, one can try to recover it at the second order. This strategy was used for example by Cerpa and Crépeau in [13] on a Korteweg De Vries equation and adapted on the considered bilinear Schrödinger equation (1.1) by Beauchard and the author in [8]. Let, for $j \in \{1,2\}$,

$$\begin{cases} i\partial_t \xi^j = -\partial^2_{xx}\xi^j - v(t)\mu(x)\Psi^j - w(t)\mu(x)\Phi_j, & (t,x) \in (0,T)\times(0,1), \\ \xi^j(t,0) = \xi^j(t,1) = 0, & t \in (0,T), \\ \xi^j(0,x) = 0, & x \in (0,1). \end{cases}$$

The main idea of this strategy is to exploit a rotation phenomenon when the control is turned off. However, as proved in [8, Lemma 4], there is no rotation phenomenon on the diagonal directions $\langle \xi^j(T), \Phi_j(T)\rangle$ and this power series expansion strategy cannot be applied to this situation.
Thus, the local exact controllability results in this article are proved using Coron's return method. This strategy, detailed in [23], relies on finding a reference trajectory of the non linear control system with suitable origin and final positions such that the linearized system around this reference trajectory is controllable. Then, the inverse mapping theorem allows to prove local exact controllability.
As the Schrödinger equation is not time reversible, the design of the reference trajectory $(\psi^1_{ref}, \dots, \psi^N_{ref}, u_{ref})$ is not straightforward. The reference control u_{ref} is designed in two steps. The first step is to impose restrictive condition on u_{ref} on an arbitrary time interval $(0,\varepsilon)$ in order to ensure the controllability of the linearized system. Then, u_{ref} is designed on (ε, T^*) such that the reference trajectory at the final time coincides with the target. For example, to prove Theorem 1.5, the reference trajectory is designed such that

$$\big(\psi^1_{ref}(T^*), \psi^2_{ref}(T^*), \psi^3_{ref}(T^*)\big) = e^{i\theta}\big(\varphi_1, \varphi_2, \varphi_3\big). \tag{1.4}$$

1.3 Structure of the article

This article is organized as follows. We recall, in Section 2, well posedness results.
To emphasize the ideas developed in this article, we start by proving Theorem 1.5. Section 3 is devoted to the construction of the reference trajectory. In Subsection 4.1, we prove the controllability of the linearized system around the reference trajectory. In Subsection 4.2, using regularity results of [6], we conclude the return method thanks to an inverse mapping argument.
In Section 5, we adapt the construction of the reference trajectory for two equations leading to Theorems 1.2 and 1.3.
Finally, Section 6 is devoted to non controllability results and the proofs of Theorems 1.1 and 1.4.

1.4 A review of previous results

Let us recall some previous results about the controllability of Schrödinger equations. In [3], Ball, Marsden and Slemrod proved a negative result for infinite dimensional bilinear control systems. The adaptation of this result to Schrödinger equations, by Turinici [45], proves that the reachable set with L^2 controls has an empty interior in $\mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$. Although this is a negative result it does not prevent controllability in more regular spaces.

Actually, in [4], Beauchard proved local exact controllability in H^7 using Nash-Moser theorem for a one dimensional model. The proof of this result was simplified, by Beauchard and Laurent in [6], by exhibiting a regularizing effect allowing to apply the classical inverse mapping theorem. In [5], Beauchard and Coron also proved exact controllability between eigenstates for a particle in a moving potential well.

Using stabilization techniques and Lyapunov functions, Nersesyan proved in [42] that Beauchard and Laurent's result holds globally in $H^{3+\varepsilon}$. Other stabilization results on similar models were obtained in [7, 38, 41, 9, 40] by Mirrahimi, Beauchard, Nersesyan and the author.

Unlike exact controllability, approximate controllability results have been obtained for Schrödinger equations on multidimensional domains. In [14], Chambrion, Mason, Sigalotti and Boscain proved approximate controllability in L^2, thanks to geometric technics on the Galerkin approximation both for the wave function and density matrices. These results were extended to stronger norms in [12] by Boussaid, Caponigro and Chambrion. Approximate controllability in more regular spaces (containing H^3) were obtained by Nersesyan and Nersisyan [43] using exact controllability in infinite time. Approximate controllability has also been obtained by Ervedoza and Puel in [26] on a model of trapped ions.

Simultaneous exact controllability of quantum particles has been obtained on a finite dimensional model in [46] by Turinici and Rabitz. Their model uses specific orientation of the molecules and their proof relies on iterated Lie brackets. In addition to the results of [14], simultaneous approximate controllability was also studied in [15] by Chambrion and Sigalotti. They used controllability of the Galerkin approximations for a model of different particles with the same control operator and a model of identical particles with different control operators. These simultaneous approximate controllability results are valid regardless of the number of particles considered.

Finally, let us give some details about the return method. This idea of designing a reference trajectory such that the linearized system is controllable was developed by Coron in [18] for a stabilization problem. It was then successfully used to prove exact controllability for various systems : Euler equations in [19, 28, 30] by Coron and Glass, Navier-Stokes equations in [20, 27, 17, 24] by Coron, Fursikov, Imanuvilov, Chapouly and Guerrero, Bürgers equations in [34, 32, 16] by Horsin, Glass, Guerrero and Chapouly and many other models such as [21, 29, 31, 25]. This method was also used for a bilinear Schrödinger equation in [4] by Beauchard.

The question of simultaneous exact controllability for linear PDE is already present in the book [37] by Lions. He considered the case of two wave equations with different boundary controls. This was later extended to other systems by Avdonin, Tucsnak, Moran and Kapitonov in [2, 1, 35].

To conclude, the question of impossibility of certain motions in small time,

at stake in this article, for bilinear Schrödinger equations was studied in [22, 8] by Coron, Beauchard and the author.

2 Well posedness

First, we recall the well posedness of the considered Schrödinger equation with a source term which proof is in [6, Proposition 2]. Consider

$$\begin{cases} i\partial_t \psi(t,x) = -\partial^2_{xx}\psi(t,x) - u(t)\mu(x)\psi(t,x) - f(t,x), & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1). \end{cases} \tag{2.1}$$

Proposition 2.1. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T>0$, $\psi_0 \in H^3_{(0)}(0,1)$, $u \in L^2((0,T),\mathbb{R})$ and $f \in L^2((0,T), H^3 \cap H^1_0)$. There exists a unique weak solution of (2.1), i.e. a function $\psi \in C^0([0,T], H^3_{(0)})$ such that the following equality holds in $H^3_{(0)}((0,1),\mathbb{C})$ for every $t \in [0,T]$,*

$$\psi(t) = e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau) + f(\tau)]d\tau.$$

Moreover, for every $R>0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|u\|_{L^2(0,T)} < R$, then this weak solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C\Big(\|\psi_0\|_{H^3_{(0)}} + \|f\|_{L^2((0,T),H^3\cap H^1_0)}\Big).$$

If $f \equiv 0$, then

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \quad \forall t \in [0,T].$$

3 Construction of the reference trajectory for three equations

The goal of this section is the design of the following family of reference trajectories.

Theorem 3.1. *Let $N=3$. Let $\mu \in H^3((0,1),\mathbb{C})$ satisfy Hypothesis 1.1 and $5\langle\mu\varphi_1,\varphi_1\rangle - 8\langle\mu\varphi_2,\varphi_2\rangle + 3\langle\mu\varphi_3,\varphi_3\rangle \neq 0$. Let $T_1>0$ be arbitrary, $\varepsilon \in (0,T_1)$ and $\varepsilon_1 \in (\frac{\varepsilon}{2},\varepsilon)$. There exist $\overline{\eta}>0$, $C>0$ such that for every $\eta \in (0,\overline{\eta})$, there exist $T^* = T^*(\eta) > T_1$, $\theta = \theta(\eta) \in \mathbb{R}$ and $u^\eta_{ref} \in L^2((0,T^*),\mathbb{R})$ with $||u^\eta_{ref}||_{L^2(0,T^*)} \leq C\eta$ such that the associated solution $\big(\psi^1_{ref}, \psi^2_{ref}, \psi^3_{ref}\big)$ of (1.2)-(1.3) satisfies*

$$\begin{aligned} \langle \mu\psi^1_{ref}(\varepsilon_1), \psi^1_{ref}(\varepsilon_1)\rangle &= \langle\mu\varphi_1,\varphi_1\rangle + \eta, \\ \langle \mu\psi^2_{ref}(\varepsilon_1), \psi^2_{ref}(\varepsilon_1)\rangle &= \langle\mu\varphi_2,\varphi_2\rangle, \\ \langle \mu\psi^3_{ref}(\varepsilon_1), \psi^3_{ref}(\varepsilon_1)\rangle &= \langle\mu\varphi_3,\varphi_3\rangle, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \langle \mu\psi^1_{ref}(\varepsilon), \psi^1_{ref}(\varepsilon)\rangle &= \langle\mu\varphi_1,\varphi_1\rangle, \\ \langle \mu\psi^2_{ref}(\varepsilon), \psi^2_{ref}(\varepsilon)\rangle &= \langle\mu\varphi_2,\varphi_2\rangle + \eta, \\ \langle \mu\psi^3_{ref}(\varepsilon), \psi^3_{ref}(\varepsilon)\rangle &= \langle\mu\varphi_3,\varphi_3\rangle, \end{aligned} \tag{3.2}$$

and

$$\big(\psi^1_{ref}(T^*), \psi^2_{ref}(T^*), \psi^3_{ref}(T^*)\big) = e^{i\theta}\big(\varphi_1, \varphi_2, \varphi_3\big). \tag{3.3}$$

Remark 3.1. For any $T \geq 0$, u^η_{ref} is extended by zero on (T^*, T^*+T). Thus, there exists $C > 0$ such that, $||u^\eta_{ref}||_{L^2(0,T^*+T)} \leq C\eta$, (3.1), (3.2) are satisfied and

$$\big(\psi^1_{ref}(T^*+T), \psi^2_{ref}(T^*+T), \psi^3_{ref}(T^*+T)\big) = e^{i\theta}\big(\Phi_1(T), \Phi_2(T), \Phi_3(T)\big).$$

Remark 3.2. The choice of a parameter η sufficiently small together with conditions (3.1) and (3.2) will be used in Section 4.1 to prove the controllability of the linearized system around the reference trajectory. The control u^η_{ref} will be designed on $(0, T_1)$ and extended by zero on (T_1, T^*).

The proof of Theorem 3.1 is divided in two steps : the construction of u^η_{ref} on $(0, \varepsilon)$ to prove (3.1) and (3.2) and then, the construction on (ε, T_1) to prove (3.3). This is what is detailed in the next subsections.

3.1 Construction on $(0, \varepsilon)$

Let $u^\eta_{ref} \equiv 0$ on $[0, \frac{\varepsilon}{2})$. We prove the following proposition.

Proposition 3.1. *Let $\mu \in H^3((0,1), \mathbb{C})$ satisfy Hypothesis 1.1. There exists $\eta^* > 0$ and a C^1 map*

$$\hat{\Gamma} : (0, \eta^*) \to L^2\left(\left(\frac{\varepsilon}{2}, \varepsilon\right), \mathbb{R}\right),$$

such that $\hat{\Gamma}(0) = 0$ and for any $\eta \in (0, \eta^*)$, the solution $\big(\psi^1_{ref}, \psi^2_{ref}, \psi^3_{ref}\big)$ of system (1.2) with control $u^\eta_{ref} := \hat{\Gamma}(\eta)$ and initial conditions $\psi^j_{ref}(\frac{\varepsilon}{2}) = \Phi_j(\frac{\varepsilon}{2})$, for $j = 1, 2, 3$, satisfies (3.1) and (3.2).

Proof of Proposition 3.1 : We consider the map

$$\begin{array}{cccc} \tilde{\Theta} : & L^2((\frac{\varepsilon}{2}, \varepsilon), \mathbb{R}) & \to & \mathbb{R}^3 \times \mathbb{R}^3 \\ & u & \mapsto & \Big(\tilde{\Theta}_1(u), \tilde{\Theta}_2(u)\Big). \end{array}$$

where

$$\tilde{\Theta}_1(u) := \big(\langle \mu\psi^j(\varepsilon_1), \psi^j(\varepsilon_1)\rangle - \langle \mu\varphi_j, \varphi_j\rangle\big)_{j=1,2,3},$$

and

$$\tilde{\Theta}_2(u) := \big(\langle \mu\psi^j(\varepsilon), \psi^j(\varepsilon)\rangle - \langle \mu\varphi_j, \varphi_j\rangle\big)_{j=1,2,3}.$$

Thanks to Proposition 2.1, $\tilde{\Theta}$ is well defined. As $\tilde{\Theta}(0) = \big((0,0,0),(0,0,0)\big)$, Proposition 3.1 is implied by application of the inverse mapping theorem to $\tilde{\Theta}$ at the point 0.

First step : C^1 regularity and differential of $\tilde{\Theta}$.
Using Proposition 2.1, it comes that $\tilde{\Theta}$ is C^1. For every $u, v \in L^2((\frac{\varepsilon}{2}, \varepsilon), \mathbb{R})$, its differential is given by

$$\mathrm{d}\tilde{\Theta}(u)(v) = \Big(\big(2\Re(\langle \mu\Psi^j(\varepsilon_1), \psi^j(\varepsilon_1)\rangle)\big)_{j=1,2,3}, \big(2\Re(\langle \mu\Psi^j(\varepsilon), \psi^j(\varepsilon)\rangle)\big)_{j=1,2,3}\Big), \tag{3.4}$$

where $\left(\Psi^1, \Psi^2, \Psi^3\right)$ is the solution of

$$\begin{cases} i\partial_t \Psi^j = -\partial_{xx}^2 \Psi^j - u(t)\mu(x)\Psi^j - v(t)\mu(x)\psi^j, & (t,x) \in (0,\varepsilon) \times (0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (0,\varepsilon), \\ \Psi^j\left(\frac{\varepsilon}{2}, x\right) = 0, & x \in (0,1), \end{cases}$$

and $\left(\psi^1, \psi^2, \psi^3\right)$ is the solution of system (1.2) with control u and initial conditions $\psi^j(\frac{\varepsilon}{2}) = \Phi_j(\frac{\varepsilon}{2})$ for $j = 1,2,3$.

Second step : invertibility of $\mathrm{d}\tilde{\Theta}(0)$.
Consider now that $u \equiv 0$. Then, for $t \in (\frac{\varepsilon}{2}, \varepsilon)$ and $j \in \{1,2,3\}$,

$$\Re\left(\langle \mu\Psi^j(t), \Phi_j(t)\rangle\right) = \sum_{k \neq j} \langle \mu\varphi_j, \varphi_k\rangle^2 \int_{\frac{\varepsilon}{2}}^t v(\tau)\sin((\lambda_k - \lambda_j)(t-\tau))\mathrm{d}\tau.$$

Let $\left((\gamma_1, \gamma_2, \gamma_3), (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)\right) \in \mathbb{R}^3 \times \mathbb{R}^3$. Thanks to Proposition A.1 (see the appendix), we can define two controls $(v_1, v_2) \in L^2((\frac{\varepsilon}{2}, \varepsilon_1), \mathbb{R}) \times L^2((\varepsilon_1, \varepsilon), \mathbb{R})$ such that for $j = 1,2,3$

$$\begin{aligned} &\int_{\frac{\varepsilon}{2}}^{\varepsilon_1} v_1(t) e^{i(\lambda_k - \lambda_j)t}\mathrm{d}t = 0, \quad \forall k \in \mathbb{N}^* \backslash \{4\}, \\ &\int_{\frac{\varepsilon}{2}}^{\varepsilon_1} v_1(t) e^{i(\lambda_4 - \lambda_j)t}\mathrm{d}t = \frac{e^{i(\lambda_4-\lambda_j)\varepsilon_1}\gamma_j}{2i\langle \mu\varphi_j, \varphi_4\rangle^2}, \\ &\int_{\varepsilon_1}^{\varepsilon} v_2(t) e^{i(\lambda_k - \lambda_j)t}\mathrm{d}t = 0, \quad \forall k \in \mathbb{N}^* \backslash \{4\}, \\ &\int_{\varepsilon_1}^{\varepsilon} v_2(t) e^{i(\lambda_4 - \lambda_j)t}\mathrm{d}t = \frac{1}{2i\langle \mu\varphi_j, \varphi_4\rangle^2}\left(e^{i(\lambda_4-\lambda_j)\varepsilon}\tilde{\gamma}_j - e^{i(\lambda_4-\lambda_j)\varepsilon_1}\gamma_j\right). \end{aligned}$$

Thus, if we define

$$v(t) := \begin{cases} v_1(t) & \text{if} \quad t \in (\frac{\varepsilon}{2}, \varepsilon_1), \\ v_2(t) & \text{if} \quad t \in (\varepsilon_1, \varepsilon), \end{cases}$$

we get that,

$$\mathrm{d}\tilde{\Theta}(0)(v) = \left((\gamma_1, \gamma_2, \gamma_3), (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)\right).$$

Finally, we can apply the inverse mapping theorem and Proposition 3.1 is proved. ■

3.2 Construction on (ε, T_1)

For any $j \in \mathbb{N}^*$, let $\mathcal{P}_j$ be the orthogonal projection of $L^2((0,1), \mathbb{C})$ onto $\mathrm{Span}_{\mathbb{C}}(\varphi_k, k \geq j+1)$ i.e.

$$\mathcal{P}_j(\psi) := \sum_{k=j+1}^{+\infty} \langle \psi, \varphi_k\rangle \varphi_k.$$

The goal of this subsection is the proof of the following proposition.

Proposition 3.2. *Let $0 < T_0 < T_f$. Let $\mu \in H^3((0,1),\mathbb{C})$ satisfy Hypothesis 1.1 and $5\langle \mu\varphi_1,\varphi_1\rangle - 8\langle \mu\varphi_2,\varphi_2\rangle + 3\langle \mu\varphi_3,\varphi_3\rangle \neq 0$. There exists $\delta > 0$ and a C^1-map*

$$\tilde{\Gamma}_{T_0,T_f} : \tilde{\mathcal{O}}_{\delta,T_0} \to L^2((T_0,T_f),\mathbb{R})$$

with

$$\tilde{\mathcal{O}}_{\delta,T_0} := \left\{ \left(\psi_0^1,\psi_0^2,\psi_0^3\right) \in (\mathcal{S}\cap H^3_{(0)}(0,1))^3 \,;\, \sum_{j=1}^{3} ||\psi_0^j - \Phi_j(T_0)||_{H^3_{(0)}} < \delta \right\},$$

such that $\tilde{\Gamma}_{T_0,T_f}\big(\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big) = 0$ and, if $(\psi_0^1,\psi_0^2,\psi_0^3) \in \tilde{\mathcal{O}}_{T_0,\delta}$, the solution $\big(\psi^1,\psi^2,\psi^3\big)$ of system (1.2) with initial conditions $\psi^j(T_0,\cdot) = \psi_0^j$, for $j = 1,2,3$, and control $u := \tilde{\Gamma}_{T_0,T_f}\big(\psi_0^1,\psi_0^2,\psi_0^3\big)$ satisfies

$$\mathcal{P}_1\big(\psi^1(T_f)\big) = \mathcal{P}_2\big(\psi^2(T_f)\big) = \mathcal{P}_3\big(\psi^3(T_f)\big) = 0, \tag{3.5}$$

$$\Im\Big(\langle\psi^1(T_f),\Phi_1(T_f)\rangle^5\overline{\langle\psi^2(T_f),\Phi_2(T_f)\rangle^8}\langle\psi^3(T_f),\Phi_3(T_f)\rangle^3\Big) = 0. \tag{3.6}$$

Remark 3.3. The conditions (3.5) and (3.6) will be used in the next subsection to prove (3.3).

Proof of Proposition 3.2 : Let us define the following space

$$X_1 := \Big\{(\phi_1,\phi_2,\phi_3) \in H^3_{(0)}((0,1),\mathbb{C})^3 \,;\, \langle\phi_j,\varphi_k\rangle = 0,\, \forall k \in \{1,\dots,j\}\Big\}.$$

We consider the following end-point map

$$\Theta_{T_0,T_f} : \quad L^2((T_0,T_f),\mathbb{R}) \times H^3_{(0)}(0,1)^3 \quad \to \quad H^3_{(0)}(0,1)^3 \times X_1 \times \mathbb{R},$$

defined by

$$\begin{aligned}\Theta_{T_0,T_f}\big(u,\psi_0^1,\psi_0^2,\psi_0^3\big) := \Big(&\psi_0^1,\psi_0^2,\psi_0^3,\, \mathcal{P}_1\big(\psi^1(T_f)\big),\mathcal{P}_2\big(\psi^2(T_f)\big),\mathcal{P}_3\big(\psi^3(T_f)\big),\\ &\Im\big(\,\langle\psi^1(T_f),\Phi_1(T_f)\rangle^5\overline{\langle\psi^2(T_f),\Phi_2(T_f)\rangle^8}\langle\psi^3(T_f),\Phi_3(T_f)\rangle^3\,\big)\Big)\end{aligned}$$

where (ψ^1,ψ^2,ψ^3) is the solution of (1.2) with initial condition $\psi^j(T_0,\cdot) = \psi_0^j$ and control u. Thus, we have

$$\Theta_{T_0,T_f}\big(0,\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big) = \big(\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0),0,0,0,0\big).$$

Proposition 3.2 is proved by application of the inverse mapping theorem to Θ_{T_0,T_f} at the point $\big(0,\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big)$.
Using the same arguments as in [6, Proposition 3], it comes that Θ_{T_0,T_f} is a C^1 map and that

$$\begin{aligned}&\mathrm{d}\Theta_{T_0,T_f}\big(0,\Phi_1(T_0),\Phi_2(T_0),\Phi_3(T_0)\big).(v,\Psi_0^1,\Psi_0^2,\Psi_0^3)\\ =&\Big(\Psi_0^1,\Psi_0^2,\Psi_0^3,\, \mathcal{P}_1\big(\Psi^1(T_f)\big),\mathcal{P}_2\big(\Psi^2(T_f)\big),\mathcal{P}_3\big(\Psi^3(T_f)\big),\\ &5\Im(\langle\Psi^1(T_f),\Phi_1(T_f)\rangle) - 8\Im(\langle\Psi^2(T_f),\Phi_2(T_f)\rangle) + 3\Im(\langle\Psi^3(T_f),\Phi_3(T_f)\rangle)\Big),\end{aligned}$$

where (Ψ^1, Ψ^2, Ψ^3) is the solution of

$$\begin{cases} i\partial_t \Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (T_0, T_f) \times (0,1),\ j \in \{1,2,3\} \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (T_0, T_f),\ j \in \{1,2,3\} \\ \Psi^j(T_0, x) = \Psi^j_0(x), & x \in (0,1),\ j \in \{1,2,3\}. \end{cases} \tag{3.7}$$

It remains to prove that $\mathrm{d}\Theta_{T_0,T_f}\big(0, \Phi_1(T_0), \Phi_2(T_0), \Phi_3(T_0)\big) : L^2((T_0,T_f),\mathbb{R}) \times H^3_{(0)}(0,1)^3 \to H^3_{(0)}(0,1)^3 \times X_1 \times \mathbb{R}$ admits a continuous right inverse.
Let $(\Psi^1_0, \Psi^2_0, \Psi^3_0) \in H^3_{(0)}(0,1)^3$, $(\psi^1_f, \psi^2_f, \psi^3_f) \in X_1$ and $r \in \mathbb{R}$. Straightforward computations lead to

$$\Psi^j(T_f) = \sum_{k=1}^{+\infty} \Big(\langle \Psi^j_0, \Phi_k(T_0)\rangle + i\langle \mu\varphi_j, \varphi_k\rangle \int_{T_0}^{T_f} v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t \Big) \Phi_k(T_f).$$

Finding $v \in L^2((T_0,T_f),\mathbb{R})$ such that

$$\begin{aligned} &\mathcal{P}_j(\Psi^j(T_f)) = \psi^j_f, \quad j \in \{1,2,3\}, \\ &\Im\big(5\langle \Psi^1(T_f), \Phi_1(T_f)\rangle - 8\langle \Psi^2(T_f), \Phi_2(T_f)\rangle + 3\langle \Psi^3(T_f), \Phi_3(T_f)\rangle\big) = r, \end{aligned}$$

is equivalent to solving the following trigonometric moment, $\forall j = 1,2,3$, $\forall k \geq j+1$

$$\begin{aligned} &\int_{T_0}^{T_f} v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t = \frac{1}{i\langle \mu\varphi_j, \varphi_k\rangle} \big(\langle \psi^j_f, \Phi_k(T_f)\rangle - \langle \Psi^j_0, \Phi_k(T_0)\rangle \big), \\ &\int_{T_0}^{T_f} v(t) \mathrm{d}t = \frac{r - \Im\big(5\langle \Psi^1_0, \Phi_1(T_0)\rangle - 8\langle \Psi^2_0, \Phi_2(T_0)\rangle + 3\langle \Psi^3_0, \Phi_3(T_0)\rangle\big)}{5\langle \mu\varphi_1, \varphi_1\rangle - 8\langle \mu\varphi_2, \varphi_2\rangle + 3\langle \mu\varphi_3, \varphi_3\rangle}. \end{aligned} \tag{3.8}$$

Using Proposition A.1 and the hypotheses on μ this ends the proof of Proposition 3.2. ∎

3.3 Proof of Theorem 3.1

Let $\delta > 0$ be the radius defined in Proposition 3.2 with $T_0 = \varepsilon$ and $T_f = T_1$. For $\eta > 0$ we define the following control

$$u^\eta_{ref}(t) := \begin{cases} 0 & \text{for } t \in (0, \frac{\varepsilon}{2}), \\ \hat{\Gamma}(\eta) & \text{for } t \in (\frac{\varepsilon}{2}, \varepsilon), \\ \tilde{\Gamma}_{\varepsilon,T_1}(\psi^1_{ref}(\varepsilon), \psi^2_{ref}(\varepsilon), \psi^3_{ref}(\varepsilon)) & \text{for } t \in (\varepsilon, T_1). \end{cases} \tag{3.9}$$

We prove that this control satisfy the conditions of Theorem 3.1.

Proof of Theorem 3.1 : The proof is decomposed into two parts. First, we prove that there exists $\overline{\eta} > 0$ such that for $\eta \in (0, \overline{\eta})$, u^η_{ref} is well defined and satisfies $||u^\eta_{ref}||_{L^2(0,T_1)} \leq C\eta$. Then, we prove the existence of $T^* > 0$ and $\theta \in \mathbb{R}$ such that if u^η_{ref} is extended by 0 on (T_1, T^*), (3.1), (3.2) and (3.3) are satisfied.

First step : u^η_{ref} is well defined.

Using Proposition 3.1, the control u^{η}_{ref} is well defined on $(0, \varepsilon)$ as soon as $\eta \in (0, \eta^*)$. Moreover, using Lipschitz property of $\hat{\Gamma}$, there exists $C(\eta^*) > 0$ such that

$$||u^{\eta}_{ref}||_{L^2(\frac{\varepsilon}{2},\varepsilon)} = ||\hat{\Gamma}(\eta) - \hat{\Gamma}(0)||_{L^2(\frac{\varepsilon}{2},\varepsilon)} \leq C(\eta^*)\eta.$$

Thanks to Proposition 2.1, there exists $C(\varepsilon) > 0$ such that if $||u||_{L^2(0,\varepsilon)} < 1$, the associated solution of (1.2)-(1.3) satisfies

$$||\psi^j(\varepsilon) - \Phi_j(\varepsilon)||_{H^3_{(0)}} \leq C(\varepsilon)||u||_{L^2(0,\varepsilon)}, \quad \text{for } j = 1, 2, 3.$$

Thus, using Proposition 3.2, if $C(\varepsilon)C(\eta^*)\eta < \frac{\delta}{3}$, we get that for $j = 1, 2, 3$,

$$||\psi^j_{ref}(\varepsilon) - \Phi_j(\varepsilon)||_{H^3_{(0)}} < \frac{\delta}{3}.$$

Thus, u^{η}_{ref} is well defined on $(0, T_1)$. Moreover, there exists $C(\delta) > 0$ such that

$$\begin{aligned}
&||u^{\eta}_{ref}||_{L^2(\varepsilon,T_1)}\\
&= ||\tilde{\Gamma}_{\varepsilon,T_1}\big(\psi^1_{ref}(\varepsilon), \psi^2_{ref}(\varepsilon), \psi^3_{ref}(\varepsilon)\big) - \tilde{\Gamma}_{\varepsilon,T_1}\big(\Phi_1(\varepsilon), \Phi_2(\varepsilon), \Phi_3(\varepsilon)\big)||_{L^2(\varepsilon,T_1)}\\
&\leq C(\delta)\sum_{j=1}^{3}||\psi^j(\varepsilon) - \Phi_j(\varepsilon)||_{H^3_{(0)}}\\
&\leq 3C(\delta)C(\varepsilon)C(\eta^*)\eta.
\end{aligned}$$

Finally, choosing

$$\overline{\eta} < \min\left(\eta^*, \frac{\delta}{3C(\varepsilon)C(\eta^*)}, \frac{1}{C(\eta^*)}\right),$$

implies that $||u^{\eta}_{ref}||_{L^2(0,T_1)} \leq C\eta$. Here and throughout this paper C denotes a positive constant that may vary each time it appears. Thanks to Proposition 3.1, it comes that (3.1) and (3.2) hold.

Second step : We prove the existence of a final time $T^* > 0$ and a global phase $\theta \in \mathbb{R}$ such that (3.3) holds.
Proposition 3.2, implies

$$\psi^j_{ref}(T_1) = \sum_{k=1}^{j}\langle\psi^j_{ref}(T_1), \Phi_k(T_1)\rangle\Phi_k(T_1), \quad \forall j = 1, 2, 3, \tag{3.10}$$

$$\Im\big(\langle\psi^1_{ref}(T_1), \Phi_1(T_1)\rangle^5\overline{\langle\psi^2_{ref}(T_1), \Phi_2(T_1)\rangle^8}\langle\psi^3_{ref}(T_1), \Phi_3(T_1)\rangle^3\big) = 0. \tag{3.11}$$

Using the invariant of the system, $\langle\psi^j_{ref}, \psi^k_{ref}\rangle \equiv \delta_{j=k}$, for $j, k \in \{1, 2, 3\}$, this leads to the existence of θ_1, $\theta_2, \theta_3 \in (-\pi, \pi]$ such that

$$\psi^j_{ref}(T_1) = e^{-i\theta_j}\Phi_j(T_1), \quad \forall j = 1, 2, 3.$$

Using (3.11), it comes that

$$\sin\big(5\theta_1 - 8\theta_2 + 3\theta_3\big) = 0.$$

Using Proposition 2.1, it comes that, up to a choice of a smaller $\overline{\eta}$,

$$5\theta_1 - 8\theta_2 + 3\theta_3 = 0. \tag{3.12}$$

Let T^* and θ be such that $T^* > T_1$ and

$$\begin{cases} T^* \equiv \dfrac{\theta_1 - \theta_2}{\lambda_2 - \lambda_1} \left[\dfrac{2}{\pi}\right], \\ \theta \equiv \dfrac{\lambda_2}{\lambda_2 - \lambda_1}\theta_1 - \dfrac{\lambda_1}{\lambda_2 - \lambda_1}\theta_2 \, [2\pi]. \end{cases}$$

This choice leads to

$$\begin{cases} \theta_1 + \lambda_1 T^* - \theta \equiv 0 \, [2\pi], \\ \theta_2 + \lambda_2 T^* - \theta \equiv 0 \, [2\pi]. \end{cases}$$

Then, using the definitions of T^* and θ and (3.12) we get

$$\begin{aligned} \theta_3 + \lambda_3 T^* - \theta &\equiv \theta_3 + \frac{\lambda_3}{\lambda_2 - \lambda_1}(\theta_1 - \theta_2) - \frac{\lambda_2}{\lambda_2 - \lambda_1}\theta_1 + \frac{\lambda_1}{\lambda_2 - \lambda_1}\theta_2 \, [2\pi] \\ &\equiv \frac{1}{3}\big(5\theta_1 - 8\theta_2 + 3\theta_3\big) \, [2\pi] \\ &\equiv 0 \, [2\pi]. \end{aligned}$$

Finally, if we extend u_{ref}^{η} by 0 on (T_1, T^*), we have that $\big(\psi_{ref}^1, \psi_{ref}^2, \psi_{ref}^3\big)$ is solution of (1.2)-(1.3) with control u_{ref}^{η} and satisfies for $j \in \{1, 2, 3\}$

$$\psi_{ref}^j(T^*) = e^{-i(\theta_j + \lambda_j T^*)}\varphi_j = e^{-i\theta}\varphi_j.$$

This ends the proof of Theorem 3.1.

■

4 Proof of Theorem 1.5

This section is dedicated to proof of Theorem 1.5. For the sake of simplicity, the proof is done in the case $T = 0$, the extension to the general case being straightforward. The proof is divided in two parts. In Subsection 4.1, the functional setting is specified and we prove the controllability of the linearized system around $(\psi_{ref}^1, \psi_{ref}^2, \psi_{ref}^3, u_{ref}^{\eta})$ for η small enough. For $j \in \{1, 2, 3\}$, let

$$\begin{cases} i\partial_t \Psi^j = -\partial_{xx}^2 \Psi^j - u_{ref}(t)\mu(x)\Psi^j - v(t)\mu(x)\psi_{ref}^j, & (t,x) \in (0,T^*) \times (0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (0,T^*), \\ \Psi^j(0,x) = 0, & x \in (0,1). \end{cases} \tag{4.1}$$

We conclude the proof of Theorem 1.5, using the linear test, in Subsection 4.2.

4.1 Controllability of the linearized system

For any $t > 0$, let

$$\begin{aligned} X_t^f := \Big\{ &(\phi^1, \phi^2, \phi^3) \in H_{(0)}^3((0,1), \mathbb{C})^3 \, ; \, \Re(\langle \phi^j, \psi_{ref}^j(t)\rangle) = 0, \text{ for } j = 1,2,3 \\ &\text{and } \langle \phi^j, \psi_{ref}^k(t)\rangle = -\overline{\langle \phi^k, \psi_{ref}^j(t)\rangle}, \text{ for } (j,k) = (2,1),(3,1),(3,2) \Big\}. \end{aligned} \tag{4.2}$$

The following proposition holds.

Proposition 4.1. *There exists $\hat{\eta} \in (0, \overline{\eta})$ such that, for any $\eta \in (0, \hat{\eta})$, if T^*, u_{ref}^{η} and $(\psi_{ref}^1, \psi_{ref}^2, \psi_{ref}^3)$ are defined as in Theorem 3.1, there exists a continuous linear map*

$$\begin{array}{rccc} L: & X_{T^*}^f & \to & L^2((0,T^*), \mathbb{R}) \\ & (\psi_f^1, \psi_f^2, \psi_f^3) & \mapsto & v \end{array}$$

such that for any $(\psi_f^1, \psi_f^2, \psi_f^3) \in X_{T^*}^f$, the solution (Ψ^1, Ψ^2, Ψ^3) of system (4.1) with control $v = L(\psi_f^1, \psi_f^2, \psi_f^3)$ satisfies

$$\big(\Psi^1(T^*), \Psi^2(T^*), \Psi^3(T^*)\big) = \big(\psi_f^1, \psi_f^2, \psi_f^3\big).$$

Before proving Proposition 4.1 we set some notations. For any $\eta \in (0, \overline{\eta})$, for any $t \in (0, T^*)$, let U_t be the propagator of the following system

$$\begin{cases} i\partial_t \psi = -\partial_{xx}^2 \psi - u_{ref}^{\eta}(t)\mu(x)\psi, & (t,x) \in (0,T^*) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T^*), \\ \psi(0,x) = \psi^0(0,x), & x \in (0,1), \end{cases}$$

i.e. $U_t \psi^0 = \psi(t)$. We also define $\mathring{U}_t$ the propagator associated to $u_{ref} \equiv 0$ i.e.

$$\mathring{U}_t \psi^0 = \sum_{k=1}^{\infty} \langle \psi^0, \varphi_k \rangle e^{-i\lambda_k t} \varphi_k.$$

For $k \geq 4$, we define $\tilde{\Phi}_k(t) := U_t \varphi_k$. For the coherence of notations, we also set $\tilde{\Phi}_j := \psi_{ref}^j$ for $j = 1,2,3$. For any $\eta \in (0, \overline{\eta})$, and any $t \in (0, T^*)$, $\big(\tilde{\Phi}_k(t)\big)_{k \in \mathbb{N}^*}$ is the image of $(\varphi_k)_{k \in \mathbb{N}^*}$ by an unitary operator. Thus, $\big(\tilde{\Phi}_k(t)\big)_{k \in \mathbb{N}^*}$ is an Hilbert basis of $L^2((0,1), \mathbb{C})$.

Proof of Proposition 4.1 : The map L will be designed on $(0, T_1)$ and extended by 0 on (T_1, T^*), where T_1 is as in Theorem 3.1.
Let

$$\mathcal{I} := \Big\{ (j,k) \in \{1,2,3\} \times \mathbb{N}^* \, ; \, k \geq j+1 \Big\} \cup \{(3,3)\},$$

and

$$\mathcal{V}_0 := \Big\{ (d^1, d^2, d^3) \in h^3(\mathbb{N}^*, \mathbb{C})^3 \, ; \, d_k^j = 0, \text{ if } (j,k) \notin \mathcal{I} \text{ and } \Re(d_3^3) = 0 \Big\}.$$

Let $R : \mathcal{I} \to \mathbb{N}$ the rearrangement such that, if $\omega_n := \lambda_k - \lambda_j$ with $n = R(j,k)$, the sequence $(\omega_n)_{n \in \mathbb{N}}$ is increasing. Notice that $0 = R(3,3)$.

First step : we prove that the directions $\langle \Psi^j(T_f), \tilde{\Phi}_k(T_f) \rangle$ for $(j,k) \in \mathcal{I}$ are controllable in any positive time T_f.
Let

$$d_{T_f} : \psi = (\psi^1, \psi^2, \psi^3) \in X_{T_f}^f \mapsto \big(d_{T_f}^1(\psi), d_{T_f}^2(\psi), d_{T_f}^3(\psi) \big) \in \mathcal{V}_0,$$

where for $j = 1,2,3$,

$$\begin{aligned} d_{T_f,k}^j(\psi) &:= \langle \psi^j, \tilde{\Phi}_k(T_f) \rangle, \quad \text{if } (j,k) \in \mathcal{I}, \\ d_{T_f,k}^j(\psi) &:= 0, \quad \text{if } (j,k) \notin \mathcal{I}. \end{aligned}$$

The next lemma ensures the controllability of the directions $\langle \Psi^j(T_f), \tilde{\Phi}_k(T_f) \rangle$ for $(j,k) \in \mathcal{I}$.

Lemma 4.1. *Let $T_f > 0$ and*

$$\begin{array}{rccc} F: & L^2((0,T_f),\mathbb{R}) & \to & \mathcal{V}_0 \\ & v & \mapsto & d_{T_f}(\Psi(T_f)) \end{array}$$

where $\Psi := \left(\Psi^1, \Psi^2, \Psi^3\right)$ is the solution of (4.1) with control v. There exists $\hat{\eta} = \hat{\eta}(T_f) \in (0, \overline{\eta})$ such that, for any $\eta \in (0, \hat{\eta})$, there exists a continuous linear map

$$\begin{array}{rccc} L_F: & \mathcal{V}_0 & \to & L^2((0,T_f),\mathbb{R}) \\ & (d^1, d^2, d^3) & \mapsto & v \end{array}$$

such that for any $(d^1, d^2, d^3) \in \mathcal{V}_0$, the function $v = L_F(d^1, d^2, d^3)$ satisfies

$$F(v) = (d^1, d^2, d^3).$$

Proof of Lemma 4.1 : The proof is based on the fact that the moment problem associated to $F(v) = d$ is close to the exponential moment problem corresponding to $u^{\eta}_{ref} \equiv 0$.

Let $v \in L^2((0,T_f),\mathbb{C})$ and $\mathring{\Psi} = (\mathring{\Psi}^1, \mathring{\Psi}^2, \mathring{\Psi}^3)$ be the solution of system (3.7). Straightforward computations lead to

$$\langle \mathring{\Psi}^j(T_f), \Phi_k(T_f) \rangle = i \langle \mu\varphi_j, \varphi_k \rangle \int_0^{T_f} v(t) e^{i(\lambda_k - \lambda_j)t} \mathrm{d}t, \quad \text{for } (j,k) \in \mathcal{I}.$$

For $n \in \mathbb{N}^*$, we define $\omega_{-n} = -\omega_n$. As proved in Proposition A.1, if $H_0 := \mathrm{Adh}_{L^2(0,T_f)}\left(\mathrm{Span}\{e^{i\omega_n \cdot}, n \in \mathbb{Z}\}\right)$, the map

$$\begin{array}{rccc} J_0: & H_0 & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & v & \mapsto & \left(\int_0^{T_f} v(t) e^{i\omega_n t} \mathrm{d}t\right)_{n\in\mathbb{Z}} \end{array}$$

is an isomorphism.

Getting back to (4.1), straightforward computations lead to

$$\langle \Psi^j(T_f), \tilde{\Phi}_k(T_f) \rangle = i \int_0^{T_f} v(t) \langle \mu\psi^j_{ref}(t), \tilde{\Phi}_k(t) \rangle \mathrm{d}t, \quad \text{for } (j,k) \in \mathcal{I}.$$

Thus, let us define

$$f_n := \frac{\overline{\langle \mu\psi^j_{ref}, \tilde{\Phi}_k \rangle}}{\langle \mu\varphi_j, \varphi_k \rangle}, \quad \text{for } (j,k) \in \mathcal{I} \text{ and } n = R(j,k),$$

and $f_{-n} := \overline{f_n}$, for $n \in \mathbb{N}^*$. We also define

$$\begin{array}{rccc} J: & L^2((0,T_f),\mathbb{C}) & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & v & \mapsto & \left(\int_0^{T_f} v(t) \overline{f_n}(t) \mathrm{d}t\right)_{n\in\mathbb{Z}} \end{array}.$$

We prove that $J_{|H_0}$ is an isomorphism. This relies on the fact that there exists $C > 0$ such that for any $v \in L^2((0,T_f),\mathbb{C})$

$$\sum_{n\in\mathbb{Z}} \Big| \int_0^{T_f} v(t)\overline{f_n}(t)\mathrm{d}t - \int_0^{T_f} v(t) e^{i\omega_n t} \mathrm{d}t \Big|^2 \leq C ||u^{\eta}_{ref}||^2_{L^2(0,T_f)} ||v||^2_{L^2(0,T_f)}. \quad (4.3)$$

Before proving this inequality, let us show that it enables to conclude the proof of Lemma 4.1.
Let $v \in L^2((0,T_f),\mathbb{C})$. As, $(e^{i\omega_n \cdot})_{n\in\mathbb{Z}}$ is a Riesz basis of H_0, [6, Proposition 19] implies that there exists $C>0$ such that

$$\sum_{n\in\mathbb{Z}} \Big| \int_0^{T_f} v(t) e^{i\omega_n t} \mathrm{d}t \Big|^2 \leq C ||v||_{L^2}^2 .$$

Together with (4.3), it implies the existence of $M>0$ such that

$$\sum_{n\in\mathbb{Z}} \Big| \int_0^{T_f} v(t) \overline{f_n}(t) \mathrm{d}t \Big|^2 \leq M ||v||_{L^2}^2 \tag{4.4}$$

Thus, J is a continuous linear map with values in $\ell^2(\mathbb{Z},\mathbb{C})$.
Estimate (4.3) also implies that for any $v \in H_0$,

$$\begin{aligned} ||J(v)-J_0(v)||_{\ell^2(\mathbb{Z},\mathbb{C})}^2 &\leq \sum_{n\in\mathbb{N}} \left| \int_0^{T_f} v(t) f_n(t) \mathrm{d}t - \int_0^{T_f} v(t) e^{i\omega_n t}\mathrm{d}t \right|^2 \\ &\quad + \sum_{n\in\mathbb{N}} \left| \int_0^{T_f} \overline{v}(t) f_n(t) \mathrm{d}t - \int_0^{T_f} \overline{v}(t) e^{-i\omega_n t}\mathrm{d}t \right|^2 \\ &\leq C ||u_{ref}^\eta||_{L^2(0,T_f)}^2 ||v||_{L^2(0,T_f)}^2 . \end{aligned}$$

This leads to

$$||J_{|H_0} - J_0||_{\mathcal{L}(H_0,\ell^2)} \leq C ||u_{ref}^\eta||_{L^2(0,T_f)} \leq C\eta .$$

As J_0 is an isomorphism, there exists $\hat{\eta} = \hat{\eta}(T_f) \in (0,\overline{\eta})$ such that, for any $\eta \in (0,\hat{\eta})$, the map $J_{|H_0}$ is an isomorphism.
Let $(d^1,d^2,d^3) \in \mathcal{V}_0$. We define $\tilde{d}_n := \dfrac{d_k^j}{\langle \mu\varphi_j,\varphi_k\rangle}$, for $(j,k)\in\mathcal{I}$ and $n=R(j,k)$, and $\tilde{d}_{-n} := \overline{d_n}$, for $n\in\mathbb{N}^*$. The map L_F defined by

$$L_F(d^1,d^2,d^3) := J_{|H_0}^{-1}(\tilde{d})$$

ends the proof of Lemma 4.1. Indeed, the construction of $\tilde{d}$ and uniqueness of solution in H_0 imply that $L_F(d^1,d^2,d^3)$ is real valued.

We now prove that (4.3) holds. Let $(j,k)\in\mathcal{I}$ and $n=R(j,k)$. Then, for any $v\in L^2((0,T_f),\mathbb{C})$,

$$\begin{aligned} &\left| \int_0^{T_f} v(t)\overline{f_n}(t)\mathrm{d}t - \int_0^{T_f} v(t) e^{i\omega_n t}\mathrm{d}t \right| \\ &= \left| \int_0^{T_f} v(t) \Big\langle \frac{\mu\psi_{ref}^j}{\langle \mu\varphi_j,\varphi_k\rangle}, \tilde{\Phi}_k(t)\Big\rangle dt - \int_0^{T_f} v(t) e^{i(\lambda_k-\lambda_j)t}\mathrm{d}t \right| \\ &= \left| \Big\langle \frac{\Psi^j(T_f)}{\langle\mu\varphi_j,\varphi_k\rangle}, \tilde{\Phi}_k(T_f)\Big\rangle - \Big\langle \frac{\mathring{\Psi}^j(T_f)}{\langle\mu\varphi_j,\varphi_k\rangle}, \Phi_k(T_f)\Big\rangle \right| \\ &\leq \left| \Big\langle \frac{\Psi^j(T_f)}{\langle\mu\varphi_j,\varphi_k\rangle}, \tilde{\Phi}_k(T_f) - \Phi_k(T_f)\Big\rangle \right| + \left| \Big\langle \frac{\Psi^j(T_f) - \mathring{\Psi}^j(T_f)}{\langle\mu\varphi_j,\varphi_k\rangle}, \Phi_k(T_f)\Big\rangle \right| \end{aligned} \tag{4.5}$$

We study each term of the previous right-hand separately. Using Hypothesis 1.1, it comes that for $j \in \{1,2,3\}$,

$$\sum_{k=1}^{\infty} \Big| \Big\langle \frac{\Psi^j(T_f) - \mathring{\Psi}^j(T_f)}{\langle \mu\varphi_j, \varphi_k \rangle}, \Phi_k(T_f) \Big\rangle \Big|^2 \leq C \big| \big| \Psi^j(T_f) - \mathring{\Psi}^j(T_f) \big| \big|^2_{H^3_{(0)}}.$$

Then, Proposition 2.1 implies that

$$\begin{aligned} ||\Psi^j(T_f) - \mathring{\Psi}^j(T_f)||_{H^3_{(0)}} &\leq C||u^\eta_{ref}\mu\mathring{\Psi}^j + v\mu(\psi^j_{ref} - \Phi_j)||_{L^2((0,T_f),H^3\cap H^1_0)} \\ &\leq C||u^\eta_{ref}||_{L^2(0,T_f)}||v||_{L^2(0,T_f)}, \end{aligned}$$

and finally,

$$\sum_{j=1}^{3}\sum_{k=1}^{\infty} \Big| \Big\langle \frac{\Psi^j(T_f) - \mathring{\Psi}^j(T_f)}{\langle \mu\varphi_j, \varphi_k \rangle}, \Phi_k(T_f) \Big\rangle \Big|^2 \leq C||u^\eta_{ref}||^2_{L^2(0,T_f)}||v||^2_{L^2(0,T_f)}. \tag{4.6}$$

We study in the same way the first term of the right-hand side of (4.5). Using Hypothesis 1.1, it comes that for $j \in \{1,2,3\}$,

$$\begin{aligned} &\sum_{k=1}^{\infty} \Big| \Big\langle \frac{\Psi^j(T_f)}{\langle \mu\varphi_j, \varphi_k \rangle}, \tilde{\Phi}_k(T_f) - \Phi_k(T_f) \Big\rangle \Big|^2 \\ &\leq C \sum_{k=1}^{\infty} \Big| k^3 \langle \Psi^j(T_f), \tilde{\Phi}_k(T_f) - \Phi_k(T_f) \rangle \Big|^2 \\ &= C \sum_{k=1}^{\infty} \Big| k^3 \langle \Psi^j(T_f), U_{T_f}\varphi_k - \mathring{U}_{T_f}\varphi_k \rangle \Big|^2 \\ &= C \sum_{k=1}^{\infty} \Big| k^3 \langle U^*_{T_f}\Psi^j(T_f) - \mathring{U}^*_{T_f}\Psi^j(T_f), \varphi_k \rangle \Big|^2 \\ &= C \, \big| \big| U^*_{T_f}\Psi^j(T_f) - \mathring{U}^*_{T_f}\Psi^j(T_f) \big| \big|^2_{H^3_{(0)}}, \end{aligned}$$

where we denoted by $U^*_{T_f}$ and $\mathring{U}^*_{T_f}$ the adjoint operators of U_{T_f} and $\mathring{U}_{T_f}$. Using unitarity, it comes that $U^*_{T_f}$ is the propagator at time T_f of system

$$\begin{cases} i\partial_t \psi = \partial^2_{xx}\psi + u^\eta_{ref}(T_f - t)\mu(x)\psi, & (t,x) \in (0,T_f) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T_f). \end{cases}$$

The same holds for $\mathring{U}^*_{T_f}$ with $u^\eta_{ref} \equiv 0$. Proposition 2.1 still holds and implies

$$\begin{aligned} &\big| \big| U^*_{T_f}\Psi^j(T_f) - \mathring{U}^*_{T_f}\Psi^j(T_f) \big| \big|_{H^3_{(0)}} \\ &\leq C \, ||u^\eta_{ref}\mu\mathring{U}^*_t\Psi^j(T_f)||_{L^2((0,T_f),H^3\cap H^1_0)} \\ &\leq C \, ||u^\eta_{ref}||_{L^2(0,T_f)}||\mathring{U}^*_t\Psi^j(T_f)||_{L^\infty((0,T_f),H^3_{(0)})} \\ &\leq C \, ||u^\eta_{ref}||_{L^2(0,T_f)}||\Psi^j(T_f)||_{H^3_{(0)}} \\ &\leq C \, ||u^\eta_{ref}||_{L^2(0,T_f)}||v||_{L^2(0,T_f)} \end{aligned}$$

Thus,

$$\sum_{j=1}^{3}\sum_{k=1}^{\infty}\Big|\Big\langle \frac{\Psi^j(T_f)}{\langle \mu\varphi_j,\varphi_k\rangle},\tilde{\Phi}_k(T_f)-\Phi_k(T_f)\Big\rangle\Big|^2 \leq C||u^\eta_{ref}||^2_{L^2(0,T_f)}||v||^2_{L^2(0,T_f)}. \quad (4.7)$$

and (4.3) holds. This ends the proof of Lemma 4.1.

□

Second step : Riesz basis and minimality.
To prove that we can also control the remaining directions, we will use the following lemmas.

Lemma 4.2. *Let $T_f > 0$ and $H := Adh_{L^2(0,T_f)}\big(Span\{f_n, n \in \mathbb{Z}\}\big)$. If $\eta < \hat{\eta}(T_f)$, then $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis of H.*

Proof of Lemma 4.2 : Using [6, Proposition 19], it comes that $(f_n)_{n\in\mathbb{Z}}$ is a Riesz basis of H if and only if there exists $C_1, C_2 > 0$ such that for any complex sequence $(a_n)_{n\in\mathbb{Z}}$ with finite support

$$C_1\left(\sum_n |a_n|^2\right)^{1/2} \leq \left|\left|\sum_n a_n f_n\right|\right|_{L^2(0,T_f)} \leq C_2\left(\sum_n |a_n|^2\right)^{1/2}.$$

Lemma 4.1 together with [10, Theorem 1] imply that there exists $C_1 > 0$ such that for any complex sequence $(a_n)_{n\in\mathbb{Z}}$ with finite support

$$\left|\left|\sum_n a_n f_n\right|\right|_{L^2(0,T_f)} \geq C_1\left(\sum_n |a_n|^2\right)^{1/2}. \quad (4.8)$$

Using again [10, Theorem 1], we get that there exists $C_2 > 0$ such that for any complex sequence $(a_n)_{n\in\mathbb{Z}}$ with finite support

$$\left|\left|\sum_n a_n f_n\right|\right|_{L^2(0,T_f)} \leq C_2\left(\sum_n |a_n|^2\right)^{1/2}, \quad (4.9)$$

if and only if, for any $g \in L^2((0,T_f),\mathbb{C})$

$$\left(\sum_{n\in\mathbb{Z}}\Big|\int_0^{T_f} g(t)\overline{f_n}(t)\mathrm{d}t\Big|^2\right)^{1/2} \leq C_2||g||_{L^2}.$$

Thus, inequality (4.4) implies that (4.9) is satisfied. Together with (4.8), it ends the proof of Lemma 4.2.

□

From now on, we consider $\hat{\eta} < \min\big(\hat{\eta}(\varepsilon/2), \hat{\eta}(T_1)\big)$ and $\eta \in (0,\hat{\eta})$ fixed for all what follows. We omit η in the notation u^η_{ref}.

Lemma 4.3. *Let $f_{j,j} := \dfrac{\langle \mu\psi^j_{ref},\psi^j_{ref}\rangle}{\langle \mu\varphi_j,\varphi_j\rangle}$, for $j \in \{1,2\}$. The family $\Xi := \big(f_n\big)_{n\in\mathbb{Z}} \cup \{f_{1,1}, f_{2,2}\}$ is minimal in $L^2((0,T_1),\mathbb{C})$.*

Proof of Lemma 4.3 : Assume that there exist $(c_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$ and $c_{1,1}$, $c_{2,2} \in \mathbb{C}$, not all being zero, such that

$$c_{1,1}f_{1,1} + c_{2,2}f_{2,2} + \sum_{n\in\mathbb{Z}} c_n f_n = 0, \quad \text{in } L^2((0,T_1),\mathbb{C}). \tag{4.10}$$

Thus,

$$c_{1,1}f_{1,1} + c_{2,2}f_{2,2} + \sum_{n\in\mathbb{Z}} c_n f_n = 0, \quad \text{in } L^2((0,\frac{\varepsilon}{2}),\mathbb{C}).$$

As $f_0 = f_{1,1} = f_{2,2} = 1$ on $(0,\frac{\varepsilon}{2})$, there exists $c\in\mathbb{C}$ such that

$$c_{1,1}f_{1,1} + c_{2,2}f_{2,2} + c_0 f_0 = c f_0, \quad \text{on } (0,\frac{\varepsilon}{2}).$$

Thus,

$$c f_0 + \sum_{n\in\mathbb{Z}^*} c_n f_n = 0, \quad \text{in } L^2((0,\frac{\varepsilon}{2}),\mathbb{C}).$$

As $\eta < \hat{\eta}(\varepsilon/2)$, Lemma 4.2 with $T_f = \varepsilon/2$ implies minimality of $(f_n)_{n\in\mathbb{Z}}$ in $L^2((0,\frac{\varepsilon}{2}),\mathbb{C})$. Thus,

$$c = 0 \quad \text{and} \quad c_n = 0, \quad \forall n \in \mathbb{Z}^*.$$

Thus, equation (4.10) imply that,

$$c_{1,1}f_{1,1} + c_{2,2}f_{2,2} + c_0 f_0 = 0, \text{ on } (0,T_1). \tag{4.11}$$

Finally, using conditions (3.1) and (3.2) in (4.11) lead to $c_{1,1} = c_{2,2} = 0$ and then $c_0 = 0$. This is a contradiction, thus the family Ξ is proved to be minimal in $L^2((0,T_1),\mathbb{C})$.

□

Third step : we recover the remaining directions.
Using [6, Proposition 18], Lemma 4.3 implies that there exists a unique biorthogonal family associated to Ξ in $\mathrm{Adh}_{L^2(0,T_1)}\big(\mathrm{Span}(\Xi)\big)$ denoted by $\big\{f'_{1,1}, f'_{2,2}, (f'_n)_{n\in\mathbb{Z}}\big\}$. This construction ensures that $f'_{1,1}$ and $f'_{2,2}$ are real valued.
Let $\psi_f \in X^f_{T^*}$ and $\tilde{\psi}_f := \big(e^{iA(T^*-T_1)}\psi_f^1, e^{iA(T^*-T_1)}\psi_f^2, e^{iA(T^*-T_1)}\psi_f^3\big)$. As u_{ref} is identically equal to 0 on (T_1,T^*), it comes that $\tilde{\psi}_f \in X^f_{T_1}$. The map L is defined by

$$L : \psi_f \in X^f_{T^*} \mapsto v \in L^2((0,T^*),\mathbb{R}),$$

where v is defined on $(0,T_1)$ by

$$v := v_0 + \sum_{j=1}^{2}\left(\Im\big(\langle\tilde{\psi}_f^j, \psi^j_{ref}(T_1)\big) - \int_0^{T_1} v_0(t)\langle\mu\psi^j_{ref}(t),\psi^j_{ref}(t)\rangle \mathrm{d}t\right) f'_{j,j}(t),$$

with $v_0 := L_F\big(d_{T_1}(\tilde{\psi}_f)\big)$ and extended by 0 on (T_1,T^*). Notice that L is linear and continuous and that as v_0, $f'_{1,1}$ and $f'_{2,2}$ are real valued so is v.
Let (Ψ^1,Ψ^2,Ψ^3) be the solution of (4.1) with control v. Using the biorthogonal properties, the definition of v_0 and Lemma 4.1 with $T_f = T_1$ we get that

$$\langle\Psi^j(T_1),\tilde{\Phi}_k(T_1)\rangle = \langle\tilde{\psi}_f^j,\tilde{\Phi}_k(T_1)\rangle, \quad \forall (j,k)\in\mathcal{I}\cup\{(1,1),(2,2)\}.$$

We check that v also controls the remaining extra-diagonal terms. Straightforward computations give

$$\langle \Psi^2(T_1), \tilde{\Phi}_1(T_1)\rangle = -\overline{\langle \Psi^1(T_1), \tilde{\Phi}_2(T_1)\rangle}.$$

Yet, by definition of v and $X^f_{T_1}$,

$$\begin{aligned}\langle \Psi^1(T_1), \psi^2_{ref}(T_1)\rangle &= \langle \tilde{\psi}^1_f, \tilde{\Phi}_2(T_1)\rangle \\ &= -\overline{\langle \tilde{\psi}^2_f, \tilde{\Phi}_1(T_1)\rangle}.\end{aligned}$$

This leads to

$$\langle \Psi^2(T_1), \tilde{\Phi}_1(T_1)\rangle = \langle \tilde{\psi}^2_f, \tilde{\Phi}_1(T_1)\rangle.$$

The same computations hold for $\langle \Psi^3(T_1), \tilde{\Phi}_1(T_1)\rangle$ and $\langle \Psi^3(T_1), \tilde{\Phi}_2(T_1)\rangle$. Thus, as $(\tilde{\Phi}_k(T_1))_{k\in\mathbb{N}^*}$ is a Hilbert basis of $L^2((0,T_1),\mathbb{C})$, it comes that

$$\big(\Psi^1(T_1), \Psi^2(T_1), \Psi^3(T_1)\big) = \big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big).$$

As v is set to zero on (T_1, T^*), this ends the proof of Proposition 4.1. ■

4.2 Controllability of the nonlinear system

In this subsection, we end the proof Theorem 1.5. We define

$$\begin{array}{rccc}\Lambda: & L^2((0,T^*),\mathbb{R}) & \to & X^f_{T^*} \\ & u & \mapsto & \big(\tilde{\mathcal{P}}_j(\psi^j(T^*))_{j=1,2,3}\big)\end{array}$$

where (ψ^1, ψ^2, ψ^3) is the solution of (1.2)-(1.3) with control u and

$$\begin{aligned}\tilde{\mathcal{P}}_j(\phi^j) :&= \phi^j - \Re\big(\langle \phi^j, \psi^j_{ref}(T^*)\rangle\big)\, \psi^j_{ref}(T^*) \\ &\quad - \sum_{k=1}^{j-1}\big(\langle \phi^j, \psi^k_{ref}(T^*)\rangle + \langle \psi^j_{ref}(T^*), \phi^k\rangle\big)\, \psi^k_{ref}(T^*).\end{aligned}$$

Thanks to this definition, Λ takes value in $X^f_{T^*}$ (defined in (4.2)) and $\Lambda(u_{ref}) = (0,0,0)$. First, we prove that we can control the projections $\tilde{\mathcal{P}}_j$. More precisely, we prove the following proposition.

Proposition 4.2. *There exists $\delta > 0$ and a C^1-map*

$$\Upsilon : \Omega_\delta \to L^2((0,T^*),\mathbb{R}),$$

with

$$\Omega_\delta := \left\{ \big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big) \in X^f_{T^*} \ ; \ \sum_{j=1}^{3} ||\tilde{\psi}^j_f||_{H^3_{(0)}} < \delta \right\}$$

such that $\Upsilon\big(0,0,0\big) = u_{ref}$ and for any $\big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big) \in \Omega_\delta$, the solution of system (1.2)-(1.3) with control $u := \Upsilon\big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big)$ satisfies

$$\big(\tilde{\mathcal{P}}_1(\psi^1(T^*)), \tilde{\mathcal{P}}_2(\psi^2(T^*)), \tilde{\mathcal{P}}_3(\psi^3(T^*))\big) = \big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big).$$

Proof of Proposition 4.2 : This proposition is proved by application of the inverse mapping theorem to Λ at the point u_{ref}. Using the same arguments as in [6, Proposition 3], it comes that Λ is C^1 and for any $v \in L^2((0,T^*),\mathbb{R})$,

$$\mathrm{d}\Lambda(u_{ref}).v = \big(\tilde{\mathcal{P}}_1(\Psi^1(T^*)), \tilde{\mathcal{P}}_2(\Psi^2(T^*)), \tilde{\mathcal{P}}_3(\Psi^3(T^*))\big),$$

where $(\Psi^j)_{j=1,2,3}$ is the solution of system (4.1) with control v. Straightforward computations lead to $\tilde{\mathcal{P}}_j(\Psi^j(T^*)) = \Psi^j(T^*)$ and thus

$$\mathrm{d}\Lambda(u_{ref}).v = \big(\Psi^1(T^*), \Psi^2(T^*), \Psi^3(T^*)\big).$$

Proposition 4.1 proves that $\mathrm{d}\Lambda(u_{ref}) : L^2((0,T^*),\mathbb{R}) \to X^f_{T^*}$ admits a continuous right inverse. Then application of the inverse mapping theorem ends the proof.

$\blacksquare$

Proof of Theorem 1.5 : Let $\tilde{\varepsilon} > 0$ and $\big(\psi^1_f, \psi^2_f, \psi^3_f\big) \in H^3_{(0)}((0,1),\mathbb{C})^3$ such that

$$\langle \psi^j_f, \psi^k_f \rangle = \delta_{j=k} \text{ and } \sum_{j=1}^{3} ||\psi^j_f - \psi^j_{ref}(T^*)||_{H^3_{(0)}} < \tilde{\varepsilon}.$$

Let

$$\big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big) := \big(\tilde{\mathcal{P}}_1(\psi^1_f), \tilde{\mathcal{P}}_2(\psi^2_f), \tilde{\mathcal{P}}_3(\psi^3_f)\big).$$

Let δ be the radius defined in Proposition 4.2. There exists $\varepsilon_0 > 0$ such that for any $\tilde{\varepsilon} \in (0, \varepsilon_0)$, $\big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big) \in \Omega_\delta$ and

$$\Re\big(\langle \psi^j_f, \psi^j_{ref}(T^*)\rangle\big) > 0, \quad \forall j \in \{1,2,3\}. \tag{4.12}$$

Let $u := \Upsilon\big(\tilde{\psi}^1_f, \tilde{\psi}^2_f, \tilde{\psi}^3_f\big)$. Let (ψ^1, ψ^2, ψ^3) be the solution of system (1.2)-(1.3) with control u. We prove that

$$\big(\psi^1(T^*), \psi^2(T^*), \psi^3(T^*)\big) = \big(\psi^1_f, \psi^2_f, \psi^3_f\big).$$

Up to a reduction of ε_0, we can assume that

$$\Re\big(\langle \psi^j(T^*), \psi^j_{ref}(T^*)\big) > 0, \quad \forall j \in \{1,2,3\}. \tag{4.13}$$

By definition of Υ and $\tilde{\mathcal{P}}_1$ it comes that

$$\psi^1(T^*) - \Re(\langle \psi^1(T^*), \psi^1_{ref}(T^*)\rangle)\psi^1_{ref}(T^*) = \psi^1_f - \Re(\langle \psi^1_f, \psi^1_{ref}(T^*)\rangle)\psi^1_{ref}(T^*).$$

Thanks to (4.12)-(4.13) and the fact that $||\psi^1(T^*)||_{L^2} = ||\psi^1_f||_{L^2}$, we get

$$\psi^1(T^*) = \psi^1_f. \tag{4.14}$$

The equality $\tilde{\mathcal{P}}_2(\psi^2(T^*)) = \tilde{\psi}^2_f$ gives

$$\begin{aligned}&\psi^2(T^*) - \langle \psi^2(T^*), \psi^1_{ref}(T^*)\rangle \psi^1_{ref}(T^*) - \Re(\langle \psi^2(T^*), \psi^2_{ref}(T^*)\rangle)\psi^2_{ref}(T^*)\\ &= \psi^2_f - \langle \psi^2_f, \psi^1_{ref}(T^*)\rangle \psi^1_{ref}(T^*) - \Re(\langle \psi^2_f, \psi^2_{ref}(T^*)\rangle)\psi^2_{ref}(T^*).\end{aligned} \tag{4.15}$$

Taking the scalar product of (4.15) with ψ_f^1, using (4.14) and the constraints $\langle \psi^2(T^*), \psi^1(T^*)\rangle = \langle \psi_f^2, \psi_f^1\rangle = 0$, it comes that

$$\begin{aligned}\langle \psi^2(T^*), \psi^1_{ref}(T^*)\rangle \langle \psi^1_{ref}(T^*), \psi_f^1\rangle + \Re\big(\langle \psi^2(T^*), \psi^2_{ref}(T^*)\rangle\big)\langle \psi^2_{ref}(T^*), \psi_f^1\rangle \\ = \langle \psi_f^2, \psi^1_{ref}(T^*)\rangle \langle \psi^1_{ref}(T^*), \psi_f^1\rangle + \Re\big(\langle \psi_f^2, \psi^2_{ref}(T^*)\rangle\big)\langle \psi^2_{ref}(T^*), \psi_f^1\rangle.\end{aligned} \tag{4.16}$$

As $||\psi^2(T^*)||_{L^2} = ||\psi_f^2||_{L^2}$, we also get

$$\begin{aligned}|\langle \psi^2(T^*), \psi^1_{ref}(T^*)\rangle|^2 + \Re\big(\langle \psi^2(T^*), \psi^2_{ref}(T^*)\rangle\big)^2 \\ = |\langle \psi_f^2, \psi^1_{ref}(T^*)\rangle|^2 + \Re\big(\langle \psi_f^2, \psi^2_{ref}(T^*)\rangle\big)^2.\end{aligned} \tag{4.17}$$

Straightforward computations prove that, up to an a priori reduction of ε_0, equalities (4.16) and (4.17) imply

$$\Re\big(\langle \psi^2(T^*), \psi^2_{ref}(T^*)\rangle\big) = \Re\big(\langle \psi_f^2, \psi^2_{ref}(T^*)\rangle\big) \tag{4.18}$$

In view of (4.16), this leads to

$$\langle \psi^2(T^*), \psi^1_{ref}(T^*)\rangle = \langle \psi_f^2, \psi^1_{ref}(T^*)\rangle. \tag{4.19}$$

Finally, using these two last inequalities in (4.15), we obtain

$$\psi^2(T^*) = \psi_f^2. \tag{4.20}$$

Using $\tilde{\mathcal{P}}_3(\psi^3(T^*)) = \tilde{\psi}_f^3$ and the exact same strategy we also get

$$\psi^3(T^*) = \psi_f^3. \tag{4.21}$$

Thus equalities (4.14), (4.20) and (4.21) end the proof of Theorem 1.5 with

$$\Gamma\big(\psi_f^1, \psi_f^2, \psi_f^3\big) := \Upsilon\big(\tilde{\mathcal{P}}_1(\psi_f^1), \tilde{\mathcal{P}}_2(\psi_f^2), \tilde{\mathcal{P}}_3(\psi_f^3)\big).$$

■

Remark 4.1. As mentioned in Remark 1.5, a slight change in the proof allows to prove Theorem 1.5 for initial conditions $(\psi_0^1, \psi_0^2, \psi_0^3)$ close enough to $(\varphi_1, \varphi_2, \varphi_3)$ satisfying

$$\langle \psi_0^j, \psi_0^k\rangle = \langle \psi_f^j, \psi_f^k\rangle, \quad \forall j,k \in \{1,2,3\}. \tag{4.22}$$

To this aim, the inverse mapping theorem is applied at the point $(u_{ref}, \varphi_1, \varphi_2, \varphi_3)$ to the map

$$\Lambda : L^2((0,T^*), \mathbb{R}) \times (\mathcal{S} \cap H^3_{(0)}(0,1))^3 \to (\mathcal{S} \cap H^3_{(0)}(0,1))^3 \times X^f_{T^*}$$

defined by

$$\Lambda\big(u, \psi_0^1, \psi_0^2, \psi_0^3\big) = \big((\psi_0^j)_{j=1,2,3}, \tilde{\mathcal{P}}_j(\psi^j(T^*))_{j=1,2,3}\big).$$

The compatibility condition (4.22) will then lead to (4.16), the conclusion being unchanged.

5 Controllability results for two equations

Theorem 1.5 leads to local exact controllability up to global phase and a global delay in the case $N = 2$. Actually the strategy we developed can be improved in this case to obtain less restrictive results, namely Theorems 1.2 and 1.3. Here, we only detail the construction of the reference trajectory, the application of the return method being very similar to Section 4.
In all this section, we consider $N = 2$. Let $T_1 > 0$ and $\varepsilon \in (0, T_1)$. As in Theorem 3.1, the reference control is designed in two steps.
Let $u \equiv 0$ on $[0, \frac{\varepsilon}{2})$. Proposition 3.1 is replaced by the following proposition.

Proposition 5.1. *There exists $\eta^* > 0$ and a C^1 map*

$$\hat{\Gamma} : (0, \eta^*) \to L^2\left(\left(\frac{\varepsilon}{2}, \varepsilon\right), \mathbb{R}\right),$$

satisfying $\hat{\Gamma}(0) = 0$ such that for any $\eta \in (0, \eta^*)$, the solution (ψ^1, ψ^2) of system (1.2) with control $u := \hat{\Gamma}(\eta)$ and initial conditions $\psi^j(\frac{\varepsilon}{2}) = \Phi_j(\frac{\varepsilon}{2})$ for $j = 1, 2$ satisfies

$$\begin{aligned}
&\langle \mu\psi^1_{ref}(\varepsilon), \psi^1_{ref}(\varepsilon)\rangle = \langle \mu\varphi_1, \varphi_1\rangle + \eta, \\
&\langle \mu\psi^2_{ref}(\varepsilon), \psi^2_{ref}(\varepsilon)\rangle = \langle \mu\varphi_2, \varphi_2\rangle.
\end{aligned}$$

As previously, this proposition will ensure controllability of the linearized system around the reference trajectory. The proof is a simple adaptation of Proposition 3.1 and is not detailed.
We now turn to two different constructions of reference trajectories on (ε, T_1), to replace Proposition 3.2.

5.1 Controllability up to a global phase in arbitrary time

Let $T > 0$ be arbitrary. Up to a reduction of ε, we assume that $T = T_1$. We prove that there exists a global phase $\theta > 0$ and a control u^η_{ref} on (ε, T) such that the associated trajectory $(\psi^1_{ref}, \psi^2_{ref})$ of (1.2)-(1.3) satisfies Proposition 5.1,

$$(\psi^1_{ref}(T), \psi^2_{ref}(T)) = e^{i\theta}(\Phi_1(T), \Phi_2(T)), \tag{5.1}$$

and $||u^\eta_{ref}||_{L^2(0,T)} \leq C\eta$.
Proposition 3.2 is replaced by the following proposition which proof is a simple adaptation of the one of Proposition 3.2 and is not detailed.

Proposition 5.2. *There exists $\delta > 0$ and a C^1-map*

$$\tilde{\Gamma} : \tilde{\mathcal{O}}_\delta \to L^2((\varepsilon, T), \mathbb{R})$$

with

$$\tilde{\mathcal{O}}_\delta := \left\{ \left(\psi^1_0, \psi^2_0\right) \in (\mathcal{S} \cap H^3_{(0)}(0,1))^2 \, ; \, \sum_{j=1}^{2} ||\psi^j_0 - \Phi_j(\varepsilon)||_{H^3_{(0)}} < \delta \right\},$$

such that $\tilde{\Gamma}\big(\Phi_1(\varepsilon), \Phi_2(\varepsilon)\big) = 0$ *and, if* $(\psi_0^1, \psi_0^2) \in \tilde{\mathcal{O}}_\delta$*, the solution* $\big(\psi^1, \psi^2\big)$ *of system (1.2) with initial conditions* $\psi^j(\varepsilon, \cdot) = \psi_0^j$*, for* $j = 1, 2$*, and control* $u := \tilde{\Gamma}\big(\psi_0^1, \psi_0^2\big)$ *satisfies*

$$\mathcal{P}_1\big(\psi^1(T)\big) = \mathcal{P}_2\big(\psi^2(T)\big) = 0, \tag{5.2}$$

$$\Im\Big(\langle \psi^1(T), \Phi_1(T)\rangle \overline{\langle \psi^2(T), \Phi_2(T)\rangle}\Big) = 0. \tag{5.3}$$

There exists $\overline{\eta} > 0$ such that for $\eta \in (0, \overline{\eta})$, the control

$$u_{ref}^\eta(t) := \begin{cases} 0 & \text{for } t \in (0, \frac{\varepsilon}{2}), \\ \hat{\Gamma}(\eta) & \text{for } t \in (\frac{\varepsilon}{2}, \varepsilon), \\ \tilde{\Gamma}(\psi_{ref}^1(\varepsilon), \psi_{ref}^2(\varepsilon)) & \text{for } t \in (\varepsilon, T), \end{cases} \tag{5.4}$$

is well defined and satisfies $||u_{ref}^\eta||_{L^2(0,T)} \leq C\eta$.
Proposition 5.3 implies that

$$\begin{aligned} &\psi_{ref}^1(T) = \langle \psi_{ref}^1(T), \Phi_1(T)\rangle \Phi_1(T), \\ &\psi_{ref}^2(T) = \langle \psi_{ref}^2(T), \Phi_1(T)\rangle \Phi_1(T) + \langle \psi_{ref}^2(T), \Phi_2(T)\rangle \Phi_2(T), \\ &\Im\big(\langle \psi_{ref}^1(T), \Phi_1(T)\rangle \overline{\langle \psi_{ref}^2(T), \Phi_2(T)\rangle}\big) = 0. \end{aligned}$$

Thus, using the invariant of the system, it comes that there exist $\theta_1, \theta_2 \in [0, 2\pi)$ such that

$$(\psi_{ref}^1(T), \psi_{ref}^2(T)) = \big(e^{-i\theta_1}\Phi_1(T), e^{-i\theta_2}\Phi_2(T)\big),$$

and

$$\theta_1 - \theta_2 \equiv 0\,[2\pi].$$

Finally, this implies that there exists $\theta \in \mathbb{R}$ such that

$$(\psi_{ref}^1(T), \psi_{ref}^2(T)) = e^{i\theta}(\Phi_1(T), \Phi_2(T)).$$

Then, application of the return method along this trajectory as in Section 4 implies Theorem 1.2.

Remark 5.1. When working up to a global phase, one can introduce a fictitious control (see [39]) and consider for $j \in \{1, 2\}$,

$$\begin{cases} i\partial_t \psi^j = -\partial_{xx}^2 \psi^j - u(t)\mu(x)\psi^j - \omega(t)\psi^j, & (t, x) \in (0, T) \times (0, 1), \\ \psi^(t, 0) = \psi^j(t, 1) = 0, & t \in (0, T). \end{cases}$$

One can prove local controllability up to a global phase by linearization of this system around the trajectory $(\Phi_1, \Phi_2, u \equiv 0, \omega \equiv 0)$.

5.2 Exact controllability in large time

We prove that there exists $T^* > 0$ and a control u_{ref}^η on (ε, T_1) such that if u_{ref}^η is extended by 0 on (T_1, T^*), the associated trajectory $(\psi_{ref}^1, \psi_{ref}^2)$ of (1.2)-(1.3) satisfies Proposition 5.1,

$$(\psi_{ref}^1(T^*), \psi_{ref}^2(T^*)) = (\varphi_1, \varphi_2), \tag{5.5}$$

and $||u_{ref}^\eta||_{L^2(0,T)} \leq C\eta$.
Proposition 3.2 is replaced by the following proposition which proof is a simple adaptation of the one of Proposition 3.2 and is not detailed.

Proposition 5.3. *There exists* $\delta > 0$ *and a* C^1*-map*

$$\tilde{\Gamma} : \tilde{\mathcal{O}}_\delta \to L^2((\varepsilon, T_1), \mathbb{R})$$

with

$$\tilde{\mathcal{O}}_\delta := \left\{ \left(\psi_0^1, \psi_0^2\right) \in (\mathcal{S} \cap H^3_{(0)}(0,1))^2 \, ; \, \sum_{j=1}^{2} ||\psi_0^j - \Phi_j(\varepsilon)||_{H^3_{(0)}} < \delta \right\},$$

such that $\tilde{\Gamma}\big(\Phi_1(\varepsilon), \Phi_2(\varepsilon)\big) = 0$ *and, if* $(\psi_0^1, \psi_0^2) \in \tilde{\mathcal{O}}_\delta$*, the solution* $\big(\psi^1, \psi^2\big)$ *of system (1.2) with initial conditions* $\psi^j(\varepsilon, \cdot) = \psi_0^j$*, for* $j = 1, 2$*, and control* $u := \tilde{\Gamma}\big(\psi_0^1, \psi_0^2\big)$ *satisfies*

$$\mathcal{P}_1\big(\psi^1(T_1)\big) = \mathcal{P}_2\big(\psi^2(T_1)\big) = 0, \tag{5.6}$$

$$\Im\Big(\langle \psi^1(T_1), \Phi_1(T_1)\rangle^4 \overline{\langle \psi^2(T_1), \Phi_2(T_1)\rangle}\Big) = 0. \tag{5.7}$$

There exists $\overline{\eta} > 0$ such that for $\eta \in (0, \overline{\eta})$, the control

$$u^\eta_{ref}(t) := \begin{cases} 0 & \text{for } t \in (0, \frac{\varepsilon}{2}), \\ \hat{\Gamma}(\eta) & \text{for } t \in (\frac{\varepsilon}{2}, \varepsilon), \\ \tilde{\Gamma}(\psi^1_{ref}(\varepsilon), \psi^2_{ref}(\varepsilon)) & \text{for } t \in (\varepsilon, T_1), \end{cases} \tag{5.8}$$

is well defined and satisfies $||u^\eta_{ref}||_{L^2(0,T_1)} \leq C\eta$.
Proposition 5.3 implies the existence of θ_1, $\theta_2 \in [0, 2\pi)$ such that

$$\begin{aligned} (\psi^1_{ref}(T_1), \psi^2_{ref}(T_1)) &= \big(e^{-i\theta_1}\Phi_1(T_1), e^{-i\theta_2}\Phi_2(T_1)\big), \\ 4\theta_1 - \theta_2 &\equiv 0 \, [2\pi]. \end{aligned}$$

Let $T^* > T_1$ be such that

$$\theta_1 + \lambda_1 T^* \equiv 0 \, [2\pi]$$

This choice leads to

$$\theta_2 + \lambda_2 T^* \equiv 4\big(\theta_1 + \lambda_1 T^*\big) \equiv 0 \, [2\pi].$$

Finally, if we extend u^η_{ref} by 0 on (T_1, T^*), we have that $(\psi^1_{ref}, \psi^2_{ref})$ is solution of (1.2)-(1.3) with control u^η_{ref} and satisfies

$$\psi^j_{ref}(T^*) = e^{-i(\theta_j + \lambda_j T^*)}\varphi_j = \varphi_j.$$

Then, application of the return method along this trajectory as in Section 4 implies Theorem 1.3.

6 Non controllability results in small time

The goal of this section is the proof of Theorems 1.1 and 1.4.

6.1 Heuristic of non controllability

We follow the strategy developed in [8] to study system (1.2) in the case $N = 2$. Using power series expansion, we consider for $j \in \{1, 2\}$,

$$u = 0 + \varepsilon v + \varepsilon^2 w,$$
$$\psi^j = \Phi_j + \varepsilon\Psi^j + \varepsilon^2\xi^j + o(\varepsilon^2).$$

This leads to the following systems, for $j \in \{1, 2\}$,

$$\begin{cases} i\partial_t\Psi^j = -\partial^2_{xx}\Psi^j - v(t)\mu(x)\Phi_j, & (t,x) \in (0,T) \times (0,1), \\ \Psi^j(t,0) = \Psi^j(t,1) = 0, & t \in (0,T), \\ \Psi^j(0,x) = 0, & x \in (0,1), \end{cases} \tag{6.1}$$

and

$$\begin{cases} i\partial_t\xi^j = -\partial^2_{xx}\xi^j - v(t)\mu(x)\Psi_j - w(t)\mu(x)\Phi_j, & (t,x) \in (0,T) \times (0,1), \\ \xi^j(t,0) = \xi^j(t,1) = 0, & t \in (0,T), \\ \xi^j(0,x) = 0, & x \in (0,1). \end{cases} \tag{6.2}$$

Let us try to reach

$$\big(\psi^1(T), \psi^2(T)\big) = \Big(\Phi_1(T), (\sqrt{1-\delta^2} + i\alpha\delta)\Phi_2(T)\Big), \tag{6.3}$$

with $\delta > 0$ and α defined in Theorem 1.1 from $(\psi^1(0), \psi^2(0)) = (\varphi_1, \varphi_2)$. Condition (6.3) imposes $\Psi^1(T) = 0$ i.e.

$$v \in V_T := \left\{ v \in L^2((0,T),\mathbb{R})\,;\, \int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}\mathrm{d}t = 0,\ \forall k \in \mathbb{N}^* \right\}.$$

Let us define the following quadratic forms, for $j \in \{1, 2\}$, associated to the second order

$$\begin{aligned} Q_{T,j}(v) : &= \Im\big(\langle\xi^j(T), \Phi_j(T)\rangle\big) \\ &= \int_0^T v(t)\int_0^t v(\tau)\left(\sum_{k=1}^{+\infty}\langle\mu\varphi_j, \varphi_k\rangle^2 \sin((\lambda_k - \lambda_j)(t-\tau))\right)\mathrm{d}\tau\mathrm{d}t, \end{aligned}$$

and

$$Q_T(v) := \langle\mu\varphi_1, \varphi_1\rangle Q_{T,2}(v) - \langle\mu\varphi_2, \varphi_2\rangle Q_{T,1}(v). \tag{6.4}$$

The following proposition states that in time small enough, the quadratic form Q_T has a sign on V_T.

Proposition 6.1. *Assume that μ satisfies Hypothesis 1.2. Then, there exists $T_* > 0$ such that for any $T \in (0, T_*)$, for any $v \in V_T\backslash\{0\}$,*

$$\mathcal{A}Q_T(v) < 0.$$

Proof of Proposition 6.1 : Let $v \in V_T$ and $s : t \in (0,T) \mapsto \int_0^t v(\tau)\mathrm{d}\tau$. Performing integrations by part, we define a new quadratic form

$$\mathcal{Q}_{T,j}(s) := -\langle(\mu')^2\varphi_j, \varphi_j\rangle\int_0^T s(t)^2\mathrm{d}t + \int_0^T s(t)\int_0^t s(\tau)h_j(t-\tau)\mathrm{d}\tau\mathrm{d}t = Q_{T,j}(v), \tag{6.5}$$

where $h_j : t \mapsto \sum_{k=1}^{+\infty}(\lambda_k - \lambda_j)^2\langle \mu\varphi_j, \varphi_k\rangle^2 \sin((\lambda_k - \lambda_j)t) \in C^0(\mathbb{R}, \mathbb{R})$. Thus, if we define

$$\mathcal{Q}_T(s) := \langle \mu\varphi_1, \varphi_1\rangle \mathcal{Q}_{T,2}(s) - \langle \mu\varphi_2, \varphi_2\rangle \mathcal{Q}_{T,1}(s), \tag{6.6}$$

we get that

$$Q_T(v) = \mathcal{Q}_T(s) = -\mathcal{A}||s||^2_{L^2} + \int_0^T s(t)\int_0^t s(\tau)h(t-\tau)\mathrm{d}\tau\mathrm{d}t,$$

with

$$h := \langle \mu\varphi_1, \varphi_1\rangle h_2 - \langle \mu\varphi_2, \varphi_2\rangle h_1 \in C^0(\mathbb{R}, \mathbb{R}).$$

We can assume, without loss of generality, that $\mathcal{A} > 0$. Thus, there exists $C = C(\mu) > 0$ such that

$$Q_T(v) \leq \big(-\mathcal{A} + CT\big)||s||^2_{L^2}. \tag{6.7}$$

Thus, we conclude the proof by choosing $T_* < \frac{\mathcal{A}}{C}$.

■

Remark 6.1. This Proposition indicates that, in small time, there are target that cannot be reached. However, using the theory of Legendre form (see e.g. [33, 11]), we can prove that Q_T lacks coercivity in $L^2((0,T), \mathbb{R})$. This is why we work directly with the quadratic form $\mathcal{Q}_T$ adapted to the auxiliary system defined in Subsection 6.2.

Remark 6.2. This strategy is only valid for small time and we do not know if this quadratic form changes sign in time large enough on V_T. Following the strategy of [8], this would imply local exact controllability in large time but it is an open question.

6.2 Auxiliary system

We consider for $j \in \{1, \dots, N\}$

$$\begin{cases} i\partial_t\tilde{\psi}^j = -\partial^2_{xx}\tilde{\psi}^j - is(t)\big(2\mu'(x)\partial_x\tilde{\psi}^j + \mu''(x)\tilde{\psi}^j\big) + s(t)^2\mu'(x)^2\tilde{\psi}^j, \\ \tilde{\psi}^j(t,0) = \tilde{\psi}^j(t,1) = 0. \end{cases} \tag{6.8}$$

and initial conditions

$$\tilde{\psi}^j(0,\cdot) = \varphi_j. \tag{6.9}$$

This control system is derived from system (1.2)-(1.3) by setting

$$\psi^j(t,x) = \tilde{\psi}^j(t,x)e^{is(t)\mu(x)} \text{ with } s(t) := \int_0^t u(\tau)\mathrm{d}\tau. \tag{6.10}$$

The well posedness of the Cauchy problem with a source term associated to system (6.8)-(6.9) may be proved similarly to Proposition 2.1 (see [8]). More precisely, consider

$$\begin{cases} i\partial_t\tilde{\psi} = -\partial^2_{xx}\tilde{\psi} - is(t)\big(2\mu'(x)\partial_x\tilde{\psi} + \mu''(x)\tilde{\psi}\big) + s(t)^2\mu'(x)^2\tilde{\psi} + f(t,x), \\ \tilde{\psi}(t,0) = \tilde{\psi}(t,1) = 0, \\ \tilde{\psi}(0,\cdot) = \tilde{\psi}_0. \end{cases} \tag{6.11}$$

We get the following proposition.

Proposition 6.2. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T>0$, $\tilde{\psi}_0 \in H^1_0(0,1)$, $s \in L^2((0,T),\mathbb{R})$ and $f \in L^2((0,T),H^1(0,1))$. There exists a unique weak solution $\tilde{\psi} \in C^0([0,T],H^1_0)$ of (6.11). Moreover, for every $R>0$, there exists $C=C(T,\mu,R)>0$ such that, if $\|s\|_{L^2(0,T)}<R$, then this weak solution satisfies*

$$\|\tilde{\psi}\|_{C^0([0,T],H^1_0)} \leqslant C\Big(\|\tilde{\psi}_0\|_{H^1_0}+\|f\|_{L^2((0,T),H^1)}\Big).$$

If $f\equiv 0$, then

$$\|\tilde{\psi}(t)\|_{L^2(0,1)}=\|\tilde{\psi}_0\|_{L^2(0,1)}, \quad \forall t\in[0,T].$$

6.3 Non exact controllability in arbitrary time with $N=2$.

In this subsection, we consider system (1.2) with $N=2$ and prove Theorem 1.1. This result is a corollary of the following theorem for the auxiliary system.

Theorem 6.1. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 1.2 hold. Let $T_*>0$ be as in Proposition 6.1 and $\alpha\in\{-1,1\}$ as in Theorem 1.1. For any $T<T_*$, there exists $\varepsilon>0$ such that for every $s\in L^2((0,T),\mathbb{R})$ with $||s||_{L^2}<\varepsilon$, the solution of system (6.8)-(6.9) satisfies*

$$\big(\tilde{\psi}^1(T),\tilde{\psi}^2(T)\big)\neq\Big(\Phi_1(T)e^{i\theta\mu},\Big(\sqrt{1-\delta^2}+i\alpha\delta\Big)\,\Phi_2(T)e^{i\theta\mu}\Big),\ \forall\delta>0,\forall\theta\in\mathbb{R}.$$

Before getting into the proof of Theorem 6.1, we prove that it implies Theorem 1.1.

Proof of Theorem 1.1 : Let $T<T_*$ and $\varepsilon>0$ defined by Theorem 6.1. Let $u\in L^2((0,T),\mathbb{R})$ be such that

$$\left(\int_0^T\left|\int_0^t u(\tau)\mathrm{d}\tau\right|^2\mathrm{d}t\right)^{1/2}<\varepsilon.$$

Assume by contradiction that

$$\big(\psi^1(T),\psi^2(T)\big)=\Big(\Phi_1(T),\Big(\sqrt{1-\delta^2}+i\alpha\delta\Big)\,\Phi_2(T)\Big),$$

for some $\delta>0$. Let s and $\tilde{\psi}^j$ be defined by (6.10). Then $||s||_{L^2}<\varepsilon$ and $\tilde{\psi}^j$ is solution of (6.8)-(6.9) and satisfies

$$\big(\tilde{\psi}^1(T),\tilde{\psi}^2(T)\big)=\Big(\Phi_1(T)e^{-is(T)\mu},\Big(\sqrt{1-\delta^2}+i\alpha\delta\Big)\,\Phi_2(T)e^{-is(T)\mu}\Big).$$

Thanks to Theorem 6.1, this is impossible.

■

Proof of Theorem 6.1 : Without loss of generality, we assume that $\mathcal{A}>0$.

First step : we prove that $-\mathcal{Q}_T$ is coercive for $T<T_*$.
Using (6.7) and the fact that $T_*<\frac{\mathcal{A}}{C}$, we get that there exists $C_*>0$ such that for $T<T_*$

$$\mathcal{Q}_T(s)\leq -C_*||s||^2_{L^2},\quad \forall s\in L^2((0,T),\mathbb{R}). \tag{6.12}$$

Second step : approximation of first and second order.
Using the first and second order approximation of (6.8) together with Proposition 6.2, the following lemma may be proved as in [8].

Lemma 6.1. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$. There exists $C = C(T) > 0$ such that for every $s \in L^2((0,T),\mathbb{R})$ with $||s||_{L^2} < 1$, the solution of (6.8)-(6.9) satisfies for all $j \in \{1,\dots,N\}$*

$$\left|\Im(\langle\tilde{\psi}^j,\Phi_j(T)\rangle) - \mathcal{Q}_{T,j}(s)\right| \leq C||s||^3_{L^2},$$
$$\left|\Im(\langle\tilde{\psi}^j,\Phi_j(T)\rangle)\right| \leq C||s||^2_{L^2}.$$

Third step : conclusion.
Let $T < T_*$. Assume by contradiction, that $\forall \varepsilon > 0$, $\exists s_\varepsilon \in L^2((0,T),\mathbb{R})$ with $||s_\varepsilon||_{L^2} < \varepsilon$ such that

$$\left(\tilde{\psi}^1_\varepsilon(T),\tilde{\psi}^2_\varepsilon(T)\right) = \left(\Phi_1(T)e^{i\theta_\varepsilon\mu}, \left(\sqrt{1-\delta_\varepsilon^2}+i\alpha\delta_\varepsilon\right)\Phi_2(T)e^{i\theta_\varepsilon\mu}\right),$$

with $\delta_\varepsilon > 0$ and $\theta_\varepsilon \in \mathbb{R}$. Notice that

$$\delta_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \quad \theta_\varepsilon \underset{\varepsilon\to 0}{\to} 0.$$

Explicit computations lead to

$$\Im\big(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle\big) = \langle\mu\varphi_1,\varphi_1\rangle\theta_\varepsilon + \underset{\varepsilon\to 0}{O}(\theta_\varepsilon^3),$$

and

$$\Im\big(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle\big) = \alpha\delta_\varepsilon + \sqrt{1-\delta_\varepsilon^2}\langle\mu\varphi_2,\varphi_2\rangle\theta_\varepsilon + \underset{\varepsilon\to 0}{O}(\theta_\varepsilon^2).$$

Thus, it comes that

$$\begin{aligned}&\langle\mu\varphi_1,\varphi_1\rangle\Im(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle) - \langle\mu\varphi_2,\varphi_2\rangle\Im(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle)\\ &= \alpha\langle\mu\varphi_1,\varphi_1\rangle\delta_\varepsilon - \langle\mu\varphi_1,\varphi_1\rangle\langle\mu\varphi_2,\varphi_2\rangle\frac{\delta_\varepsilon^2}{\sqrt{1-\delta_\varepsilon^2}+1}\theta_\varepsilon + \underset{\varepsilon\to 0}{O}(\theta_\varepsilon^2).\end{aligned}$$

Using Lemma 6.1 to estimate $\Im\big(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle\big)$, it comes that

$$\theta_\varepsilon = \underset{\varepsilon\to 0}{O}(||s_\varepsilon||^2_{L^2}).$$

Thus,

$$\begin{aligned}&\langle\mu\varphi_1,\varphi_1\rangle\Im(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle) - \langle\mu\varphi_2,\varphi_2\rangle\Im(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle)\\ &= \langle\mu\varphi_1,\varphi_1\rangle\alpha\delta_\varepsilon + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2}).\end{aligned}$$

Finally, combining this with Lemma 6.1 and (6.12), we obtain

$$\begin{aligned}0 &< \alpha\langle\mu\varphi_1,\varphi_1\rangle\delta_\varepsilon\\ &= \langle\mu\varphi_1,\varphi_1\rangle\Im(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle) - \langle\mu\varphi_2,\varphi_2\rangle\Im(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle) + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2})\\ &= \mathcal{Q}_T(s_\varepsilon) + \underset{\varepsilon\to 0}{O}(||s_\varepsilon||^3_{L^2}) + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2})\\ &\leq -C_*||s_\varepsilon||^2_{L^2} + \underset{\varepsilon\to 0}{o}(||s_\varepsilon||^2_{L^2}).\end{aligned}$$

This is impossible for ε sufficiently small. This ends the proof of Theorem 6.1. ∎

6.4 Non exact controllability up to a global phase in arbitrary time with $N=3$.

In this subsection, we consider system (1.2) with $N=3$ and prove Theorem 1.4. As previously, this result is a corollary of the following theorem for the auxiliary system.

Theorem 6.2. *Let $\mu \in H^3((0,1),\mathbb{R})$ be such that Hypothesis 1.3 hold. Let $\beta \in \{-1,1\}$ be defined as in Theorem 1.4. There exists $T_*>0$ and $\varepsilon>0$ such that for any $T<T_*$, for every $s \in L^2((0,T),\mathbb{R})$ with $||s||_{L^2}<\varepsilon$, the solution of system (1.2)-(1.3) satisfies*

$$\left(\tilde{\psi}^1(T),\tilde{\psi}^2(T),\tilde{\psi}^3(T)\right) \neq e^{i\nu}\left(\Phi_1(T)e^{i\theta\mu},\Phi_2(T)e^{i\theta\mu},\left(\sqrt{1-\delta^2}+i\beta\delta\right)\Phi_3(T)e^{i\theta\mu}\right),$$

for all $\delta>0$, $\forall \nu,\theta \in \mathbb{R}$.

The proof is similar to the one of Theorem 6.1.

Proof of Theorem 6.2 : Without loss of generality, we can assume $\mathcal{B}>0$.

First step : coercivity of the quadratic form.
We consider the following quadratic form

$$\begin{aligned}\mathcal{Q}_T(s) := & \big(\langle \mu\varphi_3,\varphi_3\rangle - \langle \mu\varphi_2,\varphi_2\rangle\big)\mathcal{Q}_{1,T}(s) + \big(\langle \mu\varphi_1,\varphi_1\rangle - \langle \mu\varphi_3,\varphi_3\rangle\big)\mathcal{Q}_{2,T}(s)\\ & + \big(\langle \mu\varphi_2,\varphi_2\rangle - \langle \mu\varphi_1,\varphi_1\rangle\big)\mathcal{Q}_{3,T}(s),\end{aligned}$$

where $\mathcal{Q}_{T,j}$ is defined as in (6.5). This is rewritten as

$$\mathcal{Q}_T(s) = -\mathcal{B}||s||_{L^2}^2 + \int_0^T s(t)\int_0^t s(\tau)h(t-\tau)\mathrm{d}\tau\mathrm{d}t,$$

with $h\in C^0(\mathbb{R},\mathbb{R})$ and $\mathcal{B}>0$. There exists $C>0$, such that

$$\mathcal{Q}_T(s) \leq \big(-\mathcal{B}+CT\big)||s||_{L^2}^2, \quad \forall s\in L^2((0,T),\mathbb{R}).$$

Thus, there exists $T_*>0$, $C_*>0$ such that for all $T<T_*$, for all $s\in L^2((0,T),\mathbb{R})$,

$$\mathcal{Q}_T(s) \leq -C_*||s||_{L^2}^2.$$

Second step : approximation of first and second order.
As Lemma 6.1 is concerned with a single equation it is still valid for $N=3$.

Third step : conclusion.
Let $T<T_*$ and assume, by contradiction, that $\forall \varepsilon>0$, $\exists s_\varepsilon \in L^2((0,T),\mathbb{R})$ with $||s_\varepsilon||_{L^2}<\varepsilon$ such that

$$\left(\tilde{\psi}^1_\varepsilon(T),\tilde{\psi}^2_\varepsilon(T),\tilde{\psi}^3_\varepsilon(T)\right) = e^{i\nu_\varepsilon}\left(\Phi_1(T)e^{i\theta_\varepsilon\mu},\Phi_2(T)e^{i\theta_\varepsilon\mu},(\sqrt{1-\delta_\varepsilon^2}+i\beta\delta_\varepsilon)\Phi_3(T)e^{i\theta_\varepsilon\mu}\right),$$

with $\nu_\varepsilon,\theta_\varepsilon\in\mathbb{R}$ and $\delta_\varepsilon>0$. Notice that,

$$\delta_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \quad \theta_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \quad e^{i\nu_\varepsilon} \underset{\varepsilon\to 0}{\to} 1.$$

Using Lemma 6.1, it comes that

$$\begin{aligned}
&\Im\big(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle\big)-\Im\big(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle\big)\\
&\quad=\big(\langle\mu\varphi_1,\varphi_1\rangle-\langle\mu\varphi_2,\varphi_2\rangle\big)\cos(\nu_\varepsilon)\theta_\varepsilon+\underset{\varepsilon\to0}{O}(\theta_\varepsilon^2)\\
&\quad=\underset{\varepsilon\to0}{O}(||s_\varepsilon||^2_{L^2}),
\end{aligned}$$

and

$$\begin{aligned}
\Im\big(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle\big)&=\sin(\nu_\varepsilon)+\langle\mu\varphi_1,\varphi_1\rangle\cos(\nu_\varepsilon)\theta_\varepsilon+\underset{\varepsilon\to0}{O}(\theta_\varepsilon^2)\\
&=\underset{\varepsilon\to0}{O}(||s_\varepsilon||^2_{L^2}).
\end{aligned}$$

This leads to

$$\begin{aligned}
\theta_\varepsilon&=\underset{\varepsilon\to0}{O}(||s_\varepsilon||^2_{L^2}),\\
\sin(\nu_\varepsilon)&=\underset{\varepsilon\to0}{O}(||s_\varepsilon||^2_{L^2}).
\end{aligned}\tag{6.13}$$

For the sake of clarity, let us denote

$$\begin{aligned}
\mathcal{I}(s):=&\big(\langle\mu\varphi_3,\varphi_3\rangle-\langle\mu\varphi_2,\varphi_2\rangle\big)\Im(\langle\tilde{\psi}^1_\varepsilon(T),\Phi_1(T)\rangle)\\
&+\big(\langle\mu\varphi_1,\varphi_1\rangle-\langle\mu\varphi_3,\varphi_3\rangle\big)\Im(\langle\tilde{\psi}^2_\varepsilon(T),\Phi_2(T)\rangle)\\
&+\big(\langle\mu\varphi_2,\varphi_2\rangle-\langle\mu\varphi_1,\varphi_1\rangle\big)\Im(\langle\tilde{\psi}^3_\varepsilon(T),\Phi_3(T)\rangle).
\end{aligned}$$

Lemma 6.1 provides the following approximation

$$|\mathcal{I}(s)-\mathcal{Q}_T(s)|\leq C||s||^3_{L^2}.$$

Using estimates (6.13), it comes that

$$\mathcal{I}(s_\varepsilon)=\beta\big(\langle\mu\varphi_2,\varphi_2\rangle-\langle\mu\varphi_1,\varphi_1\rangle\big)\cos(\nu_\varepsilon)\delta_\varepsilon+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2}).$$

Finally, for ε sufficiently small,

$$\begin{aligned}
0&<\beta\big(\langle\mu\varphi_2,\varphi_2\rangle-\langle\mu\varphi_1,\varphi_1\rangle\big)\cos(\nu_\varepsilon)\delta_\varepsilon\\
&=\mathcal{I}(s_\varepsilon)+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2})\\
&=\mathcal{Q}_T(s_\varepsilon)+\underset{\varepsilon\to0}{O}(||s_\varepsilon||^3_{L^2})+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2})\\
&\leq-C_*||s_\varepsilon||^2_{L^2}+\underset{\varepsilon\to0}{o}(||s_\varepsilon||^2_{L^2}).
\end{aligned}$$

This is impossible for ε sufficiently small. This ends the proof of Theorem 6.2. ∎

7 Conclusion, open problems and perspectives.

In this article, we have proved that the local exact controllability result of Beauchard and Laurent for a single bilinear Schrödinger equation cannot be adapted to a system of such equations with a single control. Thus, we developed a strategy based on Coron's return method to obtain controllability in arbitrary time up to a global phase or exactly up to a global delay. For three equations

local controllability up to a global phase does not even hold in small time with small controls. Thus, in this setting and under generic assumptions no local controllability result can be proved in small time if $N \geq 3$. Finally, the main result of this article is the construction of a reference trajectory and application of the return method to prove local exact controllability up to a global phase and a global delay around (Φ_1, Φ_2, Φ_3).
However our non controllability strategy is only valid for small time and we do not know if local exact controllability around the eigenstates (Φ_1, Φ_2) hold in time large enough (for two equations or more). Moreover, when Hypothesis 1.2 or 1.3 are not satisfied, we do not know if the considered quadratic forms still have a sign. Thus, the question of non controllability when these hypotheses do not hold is an open problem.
The question of controllability of four equations or more is also open and involves trickier problems : there are other directions than the diagonal ones with the same gap frequencies (e.g. $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$). A perturbation argument would be the simplest way to avoid this frequency redundancy but we would lose the rational dependency between the eigenvalues used in the construction of the reference trajectory.

Acknowledgement : The author thanks K. Beauchard for having drawn his attention to this problem and for fruitful discussions.

A Moment problems

We define the following space

$$\ell_r^2(\mathbb{N}, \mathbb{C}) := \left\{ (d_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{C})\,;\, d_0 \in \mathbb{R} \right\}.$$

In this article, we use several times the following moment problem result.

Proposition A.1. *Let T>0. Let $(\omega_n)_{n\in\mathbb{N}}$ be the increasing sequence defined by*

$$\{\omega_n\,;\, n \in \mathbb{N}\} = \{\lambda_k - \lambda_j\,;\, j \in \{1, 2, 3\}, k \geq j + 1 \text{ and } k = j = 3\}.$$

There exists a continuous linear map

$$\mathcal{L} \,:\, \ell_r^2(\mathbb{N}, \mathbb{C}) \to L^2((0, T), \mathbb{R}),$$

such that for all $d := (d_n)_{n\in\mathbb{N}}$,

$$\int_0^T \mathcal{L}(d)(t) e^{i\omega_n t} \mathrm{d}t = d_n, \quad \forall n \in \mathbb{N}.$$

Proof of Proposition A.1 : For $n \in \mathbb{N}^*$, let $\omega_{-n} := -\omega_n$. Using [36, Theorems 9.1, 9.2], it comes that for any finite interval I, there exists $C_1, C_2 > 0$, such that all finite sums

$$f(t) := \sum_n c_n e^{i\omega_n t}, \quad c_n \in \mathbb{C},$$

satisfy

$$C_1 \sum_n |c_n|^2 \leq \int_I |f(t)|^2 \mathrm{d}t \leq C_2 \sum_n |c_n|^2.$$

Let $T > 0$ and $H_0 := \text{Adh}_{L^2(0,T)}\big(\text{Span}_{\mathbb{C}}\{e^{i\omega_n \cdot}\,;\, n \in \mathbb{Z}\}\big)$. Thus, $(e^{i\omega_n \cdot})_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathcal{H}$ i.e.

$$\begin{array}{rccc} J_0 : & H_0 & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & f & \mapsto & \big(\langle f, e^{i\omega_n \cdot}\rangle\big)_{n\in\mathbb{Z}} \end{array}$$

is an isomorphism (see e.g. [6, Propositions 19, 20]). Let $d \in \ell^2_r(\mathbb{N},\mathbb{C})$. We define $\tilde{d} := (\tilde{d}_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathbb{C})$ by $\tilde{d}_n := d_n$, for $n \geq 0$ and $\tilde{d}_n := -d_{-n}$, for $n < 0$. The map $\mathcal{L}$ is defined by $\mathcal{L}(d) := J_0^{-1}(\tilde{d})$. The construction of $\tilde{d}$ and the isomorphism property ensure that $\mathcal{L}(d)$ is real valued.

■

References

[1] S.A. Avdonin and W. Moran. Simultaneous control problems for systems of elastic strings and beams. *Systems Control Lett.*, 44(2):147–155, 2001.

[2] S.A. Avdonin and M. Tucsnak. Simultaneous controllability in sharp time for two elastic strings. *ESAIM Control Optim. Calc. Var.*, 6:259–273 (electronic), 2001.

[3] J. M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.

[4] K. Beauchard. Local controllability of a 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 84(7):851–956, 2005.

[5] K. Beauchard and J.-M. Coron. Controllability of a quantum particle in a moving potential well. *J. Funct. Anal.*, 232(2):328–389, 2006.

[6] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.

[7] K. Beauchard and M. Mirrahimi. Practical stabilization of a quantum particle in a one-dimensional infinite square potential well. *SIAM J. Control Optim.*, 48(2):1179–1205, 2009.

[8] K. Beauchard and M. Morancey. Local controllability of 1d Schrödinger equations with bilinear control and minimal time. preprint, arXiv:1208.5393, 2012.

[9] K. Beauchard and V. Nersesyan. Semi-global weak stabilization of bilinear Schrödinger equations. *C. R. Math. Acad. Sci. Paris*, 348(19-20):1073–1078, 2010.

[10] R.P. Boas. A general moment problem. *Amer. J. Math.*, 63:361–370, 1941.

[11] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems.* Springer Series in Operations Research. Springer-Verlag, New York, 2000.

[12] N. Boussaid, M. Caponigro, and T. Chambrion. Weakly-coupled systems in quantum control. INRIA Nancy-Grand Est "CUPIDSE" Color program., September 2011.

[13] E. Cerpa and E. Crépeau. Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):457–475, 2009.

[14] T. Chambrion, P. Mason, M. Sigalotti, and U. Boscain. Controllability of the discrete-spectrum Schrödinger equation driven by an external field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):329–349, 2009.

[15] T. Chambrion and M. Sigalotti. Simultaneous approximate tracking of density matrices for a system of Schrödinger equations. In *Proceedings of the 48th IEEE Conference on Decision and Control*, pages 357–362, Shangai, 2009.

[16] M. Chapouly. Global controllability of nonviscous and viscous Burgers-type equations. *SIAM J. Control Optim.*, 48(3):1567–1599, 2009.

[17] M. Chapouly. On the global null controllability of a Navier-Stokes system with Navier slip boundary conditions. *J. Differential Equations*, 247(7):2094–2123, 2009.

[18] J.-M. Coron. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Systems*, 5(3):295–312, 1992.

[19] J.-M. Coron. Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(3):271–276, 1993.

[20] J.-M. Coron. On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM Contrôle Optim. Calc. Var.*, 1:35–75 (electronic), 1996.

[21] J.-M. Coron. Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations. *ESAIM Control Optim. Calc. Var.*, 8:513–554 (electronic), 2002. A tribute to J. L. Lions.

[22] J.-M. Coron. On the small-time local controllability of a quantum particle in a moving one-dimensional infinite square potential well. *C. R. Math. Acad. Sci. Paris*, 342(2):103–108, 2006.

[23] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[24] J.-M. Coron and S. Guerrero. Local null controllability of the two-dimensional Navier-Stokes system in the torus with a control force having a vanishing component. *J. Math. Pures Appl. (9)*, 92(5):528–545, 2009.

[25] J.-M. Coron, S. Guerrero, and L. Rosier. Null controllability of a parabolic system with a cubic coupling term. *SIAM J. Control Optim.*, 48(8):5629–5653, 2010.

[26] S. Ervedoza and J.-P. Puel. Approximate controllability for a system of Schrödinger equations modelling a single trapped ion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(6):2111–2136, 2009.

[27] A.V. Fursikov and O.Y. Imanuvilov. Exact controllability of the Navier-Stokes and Boussinesq equations. *Uspekhi Mat. Nauk*, 54(3(327)):93–146, 1999.

[28] O. Glass. Exact boundary controllability of 3-D Euler equation. *ESAIM Control Optim. Calc. Var.*, 5:1–44 (electronic), 2000.

[29] O. Glass. On the controllability of the Vlasov-Poisson system. *J. Differential Equations*, 195(2):332–379, 2003.

[30] O. Glass. On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)*, 9(3):427–486, 2007.

[31] O. Glass. Controllability and asymptotic stabilization of the Camassa-Holm equation. *J. Differential Equations*, 245(6):1584–1615, 2008.

[32] O. Glass and S. Guerrero. On the uniform controllability of the Burgers equation. *SIAM J. Control Optim.*, 46(4):1211–1238, 2007.

[33] M.R. Hestenes. Applications of the theory of quadratic forms in Hilbert space to the calculus of variations. *Pacific J. Math.*, 1:525–581, 1951.

[34] T. Horsin. On the controllability of the Burgers equation. *ESAIM Control Optim. Calc. Var.*, 3:83–95 (electronic), 1998.

[35] B. Kapitonov. Simultaneous exact controllability for a class of evolution systems. *Comput. Appl. Math.*, 18(2):149–161, 257, 1999.

[36] V. Komornik and P. Loreti. *Fourier series in control theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.

[37] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[38] M. Mirrahimi. Lyapunov control of a quantum particle in a decaying potential. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1743–1765, 2009.

[39] M. Mirrahimi, P. Rouchon, and G. Turinici. Lyapunov control of bilinear Schrödinger equations. *Automatica J. IFAC*, 41(11):1987–1994, 2005.

[40] M. Morancey. Explicit approximate controllability of the Schrödinger equation with a polarizability term. preprint, arXiv:1110.2860v1, 2011.

[41] V. Nersesyan. Growth of Sobolev norms and controllability of the Schrödinger equation. *Comm. Math. Phys.*, 290(1):371–387, 2009.

[42] V. Nersesyan. Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):901–915, 2010.

[43] V. Nersesyan and H. Nersisyan. Global exact controllability in infinite time of Schrödinger equation: multidimensional case. Preprint, hal-00660478, 2012.

[44] P. Rouchon. Control of a quantum particle in a moving potential well. In *Lagrangian and Hamiltonian methods for nonlinear control 2003*, pages 287–290. IFAC, Laxenburg, 2003.

[45] G. Turinici. On the controllability of bilinear quantum systems. In *Mathematical models and methods for ab initio quantum chemistry*, volume 74 of *Lecture Notes in Chem.*, pages 75–92. Springer, Berlin, 2000.

[46] G. Turinici and H. Rabitz. Optimally controlling the internal dynamics of a randomly oriented ensemble of molecules. *Phys. Rev. A*, 70:063412, Dec 2004.